To complete the proof it suffices to prove that $(\tilde{K},f)$ is a triangulation of $\tilde{X}$ (that is, that f is a homeomorphism). If v is a vertex of K, then st v, being contractible, is evenly covered by p. For $\tilde{v} \in p^{-1}(v)$ let $U_{\tilde{v}}$ be the component of $p^{-1}(\text{st } v)$ containing $\tilde{v}$. Then $p \mid U_{\tilde{v}}$ is a homeomorphism of $U_{\tilde{v}}$ onto st v. By the definition of $\tilde{K}$ and φ, $|\varphi| \mid \text{st } \tilde{v}$ is a homeomorphism of st $\tilde{v}$ onto st v for $\tilde{v} \in p^{-1}(v)$. From the commutativity of the above triangle, $f \mid \text{st } \tilde{v}$ is a homeomorphism of st $\tilde{v}$ onto $U_{\tilde{v}}$ for $\tilde{v} \in p^{-1}(v)$. Since $|\varphi|^{-1}(\text{st } v) = \cup\{\text{st } \tilde{v} \mid \tilde{v} \in p^{-1}(v)\}$, $f \mid |\varphi|^{-1}(\text{st } v)$ is a homeomorphism of $|\varphi|^{-1}(\text{st } v)$ onto $p^{-1}(\text{st } v)$. Since this is so for every vertex v of K, f is a homeomorphism of $|\tilde{K}|$ onto $\tilde{X}$. ∎

The following corollary is an interesting application of these results.

4 COROLLARY *Any subgroup of a free group is free.*

PROOF Let F be a free group. It follows from example 3.7.6 that there is a polyhedron (in fact, a wedge of 1-spheres) X with base point x_0 such that $\pi(X,x_0) \approx F$. Let F' be any subgroup of F. Under the above isomorphism F' corresponds to some subgroup $H \subset \pi(X,x_0)$. Let $p\colon \tilde{X} \to X$ be a covering projection such that $\tilde{X}$ is path connected, $p(\tilde{x}_0) = x_0$, and $p_\#\pi(\tilde{X},\tilde{x}_0) = H$. By theorem 3, $\tilde{X}$ is homeomorphic to the space of a connected graph. By corollary 3.7.5, $\pi(\tilde{X},\tilde{x}_0)$ is a free group. ∎

If K is a finite connected graph, it follows from corollary 3.7.5 that $E(K,v_0)$ is a free group on $1 - n_0 + n_1$ generators, where n_0 is the number of vertices of K and n_1 is the number of 1-simplexes of K. If $p\colon \tilde{X} \to |K|$ is a covering projection of multiplicity m, the number of q-simplexes in the corresponding triangulation $(\tilde{K},f)$ of $\tilde{X}$ (given by theorem 3) equals mn_q, where n_q is the number of q-simplexes of K. Therefore the method used to prove corollary 4 also yields the following result.

5 COROLLARY *Let F be a free group on n generators and let F' be a subgroup of F of index m. Then F' is a free group on $1 - m + mn$ generators.* ∎

We now investigate the effect on the fundamental group of the process of attaching cells. Let A be a closed subset of a space X. X is said to be *obtained from A by adjoining n-cells* $\{e_j^n\}$, where $n \ge 0$, if

(*a*) For each j, e_j^n is a subset of X.
(*b*) If $\dot{e}_j^n = e_j^n \cap A$, then for $j \neq j'$, $e_j^n - \dot{e}_j^n$ is disjoint from $e_{j'}^n - \dot{e}_{j'}^n$.
(*c*) X has a topology coherent with $\{A,e_j^n\}$ and $X = A \cup \cup_j e_j^n$.
(*d*) For each j there is a map

$$f_j\colon (E^n,S^{n-1}) \to (e_j^n,\dot{e}_j^n)$$

such that $f_j(E^n) = e_j^n$, f_j maps $E^n - S^{n-1}$ homeomorphically onto $e_j^n - \dot{e}_j^n$, and e_j^n has the topology coinduced by f_j and the inclusion map $\dot{e}_j^n \subset e_j^n$.

Note that if $n = 0$, X is the topological sum of A and a discrete space. A map f_j: $(E^n,S^{n-1}) \to (e_j^n,\dot{e}_j^n)$ satisfying condition (*d*) above is called a *characteristic map* for e_j^n, and $f_j \mid S^{n-1}\colon S^{n-1} \to A$ is called an *attaching map* for e_j^n. X is characterized by A and the collection $\{f_j \mid S^{n-1}\}$ of attaching maps. Given A and an indexed collection of maps $\{g_j\colon S^{n-1} \to A\}$, there is a space X obtained from A by attaching n-cells $\{e_j^n\}$ by the maps g_j. X is defined as the quotient space of the topological sum $\bigvee E_j^n \vee A$, where $E_j^n = E^n$ for each j, by the identifications $z \in S_j^{n-1}$ equals $g_j(z) \in A$. Then the inclusion map $(E_j^n, S_j^{n-1}) \subset (\bigvee E_j^n \vee A, \bigvee S_j^{n-1} \vee A)$ followed by the projection to (X,A) is a characteristic map $f_j\colon (E_j^n, S_j^{n-1}) \to (X,A)$ for an n-cell $e_j^n = f_j(E_j^n)$.

Following are two examples.

6 If K is a simplicial complex, $|K^q|$ is obtained from $|K^{q-1}|$ by adjoining q-cells $\{|s| \mid s \text{ is a } q\text{-simplex of } K\}$.

7 For $i = 1, 2$, or 4 let F_i be $\mathbf{R}$, $\mathbf{C}$, or $\mathbf{Q}$, respectively, and for $q \geq 0$ let $P_q(F_i)$ be the real, complex, or quaternionic projective space of dimension q. $P_q(F_i)$ is imbedded in $P_{q+1}(F_i)$ by the map $[t_0, t_1, \ldots, t_q] \to [t_0, t_1, \ldots, t_q, 0]$ for $t_j \in F_i$. Then $P_{q+1}(F_i)$ is obtained from $P_q(F_i)$ by adjoining a single $(q+1)i$-cell. If $E^{(q+1)i}$ is identified with the space $\{(t_0, t_1, \ldots, t_q) \in F_i^{q+1} \mid \sum |t_j|^2 \leq 1\}$, then a characteristic map $f\colon (E^{(q+1)i}, S^{(q+1)i-1}) \to (P_{q+1}(F_i), P_q(F_i))$ for this single cell is defined by the equation

$$f(t_0, t_1, \ldots, t_q) = [t_0, t_1, \ldots, t_q, 1 - \sum |t_j|^2]$$

8 LEMMA *Let X be obtained from A by adjoining n-cells for $n \geq 2$. Then for any point $x_0 \in A$ the inclusion map $i\colon (A,x_0) \subset (X,x_0)$ induces an epimorphism*

$$i_\#\colon \pi(A,x_0) \to \pi(X,x_0)$$

PROOF Let X be obtained from A by adjoining the n-cells $\{e_j^n\}$, and for each j let $y_j \in e_j^n - \dot{e}_j^n$ and let B_j be a neighborhood of y_j in $e_j^n - \dot{e}_j^n$ homeomorphic to E^n. Let $\omega\colon (I,\dot{I}) \to (X,x_0)$ be a closed path at x_0. We show that ω is homotopic to a path in $U = X - \{y_j\}_j$. By the compactness of I, we can subdivide I by points $0 = t_0 < t_1 < \cdots < t_n = 1$ such that for $0 \leq i < n$ either $\omega[t_i, t_{i+1}] \subset U$ or $\omega[t_i, t_{i+1}] \subset B_j$ for some j. If $\omega[t_i, t_{i+1}] \cup \omega[t_{i+1}, t_{i+2}] \subset B_j$, we can omit the point t_{i+1} from the subdivision of I to obtain another subdivision of I with the same property. Continuing in this way we can obtain a subdivision such that if $\omega[t_i, t_{i+1}] \subset B_j$, then neither $\omega[t_{i-1}, t_i]$ nor $\omega[t_{i+1}, t_{i+2}]$ is contained in B_j. It follows that $\omega(t_i) \neq y_j$ and $\omega(t_{i+1}) \neq y_j$. For each such i, because $B_j - y_j$ is path connected and B_j is simply connected, $\omega \mid [t_i, t_{i+1}]$ is homotopic rel $\{t_i, t_{i+1}\}$ to a path contained in $B_j - y_j$. Since altogether there are only a finite number of such subintervals of I, $\omega \simeq \omega'$, where $\omega'(I) \subset U$.

Because S^{n-1} is a strong deformation retract of E^n minus a point, it follows that $\dot{e}_j^n$ is a strong deformation retract of $e_j^n - y_j$. Therefore A is a strong deformation retract of U and $\omega' \simeq \omega''$, where $\omega''(I) \subset A$. Then $i_\#[\omega''] = [\omega]$. ∎

9 COROLLARY *For all $n \geq 0$, $P_n(\mathbf{C})$ and $P_n(\mathbf{Q})$ are simply connected.*

PROOF Because $P_0(\mathbf{C})$ and $P_0(\mathbf{Q})$ are each one-point spaces, the result follows by induction on q, using lemma 8 and the fact that $P_{q+1}(\mathbf{C})$ is obtained from $P_q(\mathbf{C})$ by adjoining a $2(q + 1)$-cell and $P_{q+1}(\mathbf{Q})$ is obtained from $P_q(\mathbf{Q})$ by adjoining a $4(q + 1)$-cell. ■

We want to compute the kernel of $i_\#$ for the case $n = 2$. Given any map $g: S^1 \to A$, where A is path connected, and given a point $x_0 \in A$, a normal subgroup of $\pi(A,x_0)$ is determined as follows. If $g(p_0) = x_1$ and ω is a path in A from x_0 to x_1, then $h_{[\omega]}g_\#(\pi(S^1,p_0))$ is a cyclic subgroup of $\pi(A,x_0)$, and for a different choice of ω we obtain a conjugate subgroup in $\pi(A,x_0)$. Therefore the normal subgroup of $\pi(A,x_0)$ generated by $h_{[\omega]}g_\#(\pi(S^1,p_0))$ is independent of the choice of the path ω. Similar statements apply to a collection of maps $\{g_j: S^1 \to A\}$. There is a well-defined normal subgroup of $\pi(A,x_0)$ determined by these maps.

10 THEOREM *Let A be a connected polyhedron and let X be obtained from A by attaching 2-cells to A by maps $\{g_j: S^1 \to A\}$. If N is the normal subgroup of $\pi(A,x_0)$ determined by the maps $\{g_j\}$, then*

$$i_\#: \pi(A,x_0) \to \pi(X,x_0)$$

is an epimorphism with kernel N.

PROOF By lemma 8, $i_\#$ is a surjection. Let $p: \tilde{A} \to A$ be a covering projection such that $\tilde{A}$ is path connected, $p(\tilde{x}_0) = x_0$, and $p_\#(\pi(\tilde{A},\tilde{x}_0)) = N$. Because N is normal in $\pi(A,x_0)$, p is a regular covering projection. Because N is the subgroup determined by the maps $\{g_j\}$, each map g_j lifts to a map $\tilde{g}_j: S^1 \to \tilde{A}$. Let $\tilde{X}$ be the space obtained from $\tilde{A}$ by attaching 2-cells for all the lifted maps $\{\tilde{g}_j\}$ and extend p to a map $p': \tilde{X} \to X$ such that p' maps each 2-cell of $\tilde{X}$ homeomorphically onto its corresponding 2-cell of X. Then p' is easily seen to be a covering projection.

We know from the definition of N that $i_\#(N) = 1$. Assume that $[\omega] \in \pi(A,x_0)$ is in the kernel of $i_\#$. Let $\tilde{\omega}$ be any lifting of ω in $\tilde{A}$ such that $\tilde{\omega}(0) = \tilde{x}_0$. Then $\tilde{\omega}$ is a lifting of ω in $\tilde{X}$. Because ω is null homotopic in X, $\tilde{\omega}$ is a closed path in $\tilde{X}$. Therefore $\tilde{\omega}$ is a closed path in $\tilde{A}$, and so

$$[\omega] = p_\#[\tilde{\omega}] \in N \quad ■$$

Note that for the proof of theorem 10 it was not necessary that A be a connected polyhedron. It would have been sufficient to assume A path connected, locally path connected, and semilocally 1-connected.

11 COROLLARY *For any group G there is a space X with $\pi(X,x_0) \approx G$.*

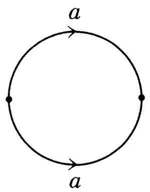
PROOF Represent G as the quotient group of a free group F and a normal subgroup N. There is a polyhedron A such that $\pi(A,x_0) \approx F$ (in fact, as in example 3.7.6, A can be taken to be a wedge of 1-spheres). For each $\lambda \in N$

let g_λ: $(S^1,p_0) \to (A,x_0)$ be a map such that $[g_\lambda]$ corresponds to λ under the isomorphism $\pi(A,x_0) \approx F$. Let X be the space obtained from A by attaching 2-cells by the maps $\{g_\lambda\}$. By theorem 10, there is an isomorphism $\pi(X,x_0) \approx G$. ■

We now specialize to the case of a surface. These are the spaces of finite two-dimensional pseudomanifolds without boundary. An *n-dimensional pseudomanifold without boundary* (or *absolute n-circuit*) is a simplicial complex K such that

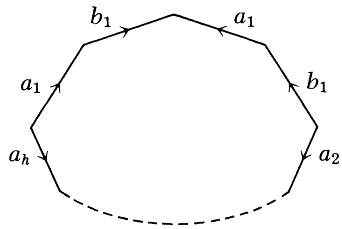
(*a*) Every simplex of K is a face of some n-simplex of K.
(*b*) Every $(n-1)$-simplex of K is the face of exactly two n-simplexes of K.
(*c*) If s and s' are n-simplexes of K, there is a finite sequence $s = s_1, s_2, \ldots, s_m = s'$ of n-simplexes of K such that s_i and s_{i+1} have an $(n-1)$-face in common for $1 \leq i < m$.

We define a *surface* to be the space of a finite two-dimensional pseudomanifold without boundary in which the star of every vertex is homeomorphic to $\mathbf{R}^2$. It can be shown[1] that every surface has a normal form consisting of a polygon in the plane with identifications of its edges. These fall into classes, those with $h \geq 0$ handles and those with k crosscaps. The surface with 0 handles is the polygon with identifications of its edges pictured as



Surface with 0 *handles*

Topologically it is homeomorphic to the 2-sphere S^2. For $h > 0$ the surface with h handles is pictured as

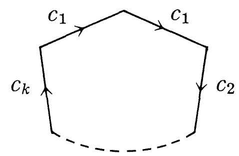


Surface with $h > 0$ *handles*

The surface with one handle is topologically the *torus*.

[1] See S. Lefschetz, *Introduction to Topology*, Princeton University Press, Princeton, N.J., 1949, and H. Seifert and W. Threlfall, *Lehrbuch der Topologie*, B. G. Teubner, Verlagsgesellschaft, Leipzig, 1934.

For $k \geq 1$, the surface with k crosscaps is pictured as



Surface with k crosscaps

The surface with one crosscap is topologically the real projective plane P^2, and the surface with two crosscaps is topologically the *Klein bottle.*

The normal form represents a surface with $h \geq 1$ handles as a wedge of $2h$ 1-spheres with a single 2-cell attached by a suitable map. If A is the wedge of $2h$ 1-spheres, then $\pi(A)$ is a free group on $2h$ generators, which generators we denote by a_i and b_i, where $1 \leq i \leq n$. If X is the surface with h handles, X is obtained from A by attaching a single 2-cell to A by a map $g\colon S^1 \to A$ such that $g_\#$ maps a generator of $\pi(S^1)$ to the element $a_1b_1a_1^{-1}b_1^{-1} \cdots a_hb_ha_h^{-1}b_h^{-1} \in \pi(A)$. Theorem 10 then provides a description of $\pi(X)$ in terms of generators and relations. Similar remarks apply to a surface with $k \geq 1$ crosscaps. The result is summarized below.

12 *The fundamental group of a surface is*

(*a*) *Trivial for the surface with no handles.*
(*b*) *A group with generators* $a_1, b_1, \ldots, a_h, b_h$ *and the single relation* $a_1b_1a_1^{-1}b_1^{-1} \cdots a_hb_ha_h^{-1}b_h^{-1} = 1$ *for a surface with* $h \geq 1$ *handles.*
(*c*) *A group with generators* $c_1, c_2, \ldots, c_k$ *and the single relation* $c_1^2c_2^2 \cdots c_k^2 = 1$ *for a surface with* $k \geq 1$ *crosscaps.* ■

EXERCISES

A TOPOLOGICAL PROPERTIES OF POLYHEDRA

1 Prove that a compact polyhedron is an absolute neighborhood retract. (*Hint:* Assume $X = |K|$ and let K be a subcomplex of a simplex s. Use induction on the number of simplexes in $s - K$ and the fact that a retract of an open subset of an absolute neighborhood retract is an absolute neighborhood retract.)

2 Give an example of a space X and closed subset $A \subset X$ such that A and X are both polyhedra but (X,A) is not a polyhedral pair.

3 Prove that an open subset of a compact polyhedron is a polyhedron. [*Hint:* Since $|K| - U$ is a G_δ, there exists a sequence of open subsets V_i of $|K|$ such that $\bigcap V_i = |K| - U$. By induction on n, construct a sequence of subdivisions K_n and subcomplexes $L_n \subset K_n$ such that (*a*) K_n is finer than the covering $\{U,V_n\}$, (*b*) L_n is the largest subcomplex of K_n such that $|L_n| \subset U$, and (*c*) K_{n+1} is a subdivision of K_n containing L_n as subcomplex. Then $L = \bigcup L_n$ is a simplicial complex such that $|L| = |K| - U$.]

4 Let Y be an n-connected space and K be a simplicial complex. Prove that any continuous map $|K| \to Y$ is homotopic to a map which sends $|K^n|$ to a single point. If f_0, f_1: $(|K|,|K^n|) \to (Y,y_0)$ are homotopic, prove that they are homotopic relative to $|K^{n-1}|$.

5 Let Y be a space which is n-connected for every n and let (X,A) be a polyhedral pair. Prove that two maps $X \to Y$ which agree on A are homotopic relative to A.

6 Prove that a polyhedron is contractible if and only if it is n-connected for every n. If it has finite dimension m, it is contractible if and only if it is m-connected.

B EXAMPLES

1 Prove that P^n is a polyhedron for all n.

2 Let K be the simplicial complex consisting of vertices $v_1, v_2, \ldots, v_p$ and simplexes $\{v_1,v_2\}, \{v_2,v_3\}, \ldots, \{v_{p-1},v_p\}$, and $\{v_p,v_1\}$ and let I be the simplicial complex with 0 and 1 as vertices and $\{0,1\}$ as 1-simplex. Then $K * I$ is a simplicial complex with vertices $v_1, \ldots, v_p$, 0, and 1. If q is an integer relatively prime to p and v_i is defined for all integers i to be equal to v_j if $i \equiv j \bmod p$, then let X be the space obtained from $|K * I|$ by identifying the 2-simplex $\{v_i,v_{i+1},0\}$ linearly with the 2-simplex $\{v_{i+q},v_{i+q+1},1\}$ for all i. Prove that X is homeomorphic to the lens space $L(p,q)$ and that X is a polyhedron.

3 Prove that the generalized lens space $L(p, q_1, \ldots, q_n)$ is a polyhedron.

4 If X and Y are polyhedra and one of them is locally compact, prove that $X * Y$ and $X \times Y$ are also polyhedra.

C PSEUDOMANIFOLDS

A simplicial complex is said to be *homogeneously n-dimensional* if every simplex is a face of some n-simplex of the complex. An *n-dimensional pseudomanifold* is a simplicial complex K such that

(*a*) K is homogeneously n-dimensional.
(*b*) Every $(n - 1)$-simplex of K is the face of at most two n-simplexes of K.
(*c*) If s and s' are n-simplexes of K, there is a finite sequence $s = s_1, s_2, \ldots, s_m = s'$ of n-simplexes of K such that s_i and s_{i+1} have an $(n - 1)$-face in common for $1 \leq i < m$.

The *boundary* of an n-dimensional pseudomanifold K, denoted by $\dot{K}$, is defined to be the subcomplex of K generated by the set of $(n - 1)$-simplexes which are faces of exactly one n-simplex of K. (If $\dot{K}$ is empty, then K is an n-dimensional pseudomanifold without boundary, as defined in Sec. 3.8.)

1 Prove that an n-simplex is an n-dimensional pseudomanifold whose boundary, as a pseudomanifold, is $\dot{s}$.

2 If K is a pseudomanifold and L is a subdivision of K, prove that L is a pseudomanifold and $\dot{L} = L \mid \dot{K}$.

3 If K is a finite 1-dimensional pseudomanifold, prove that $\dot{K}$ is either empty or consists of exactly two vertices.

4 Give an example of an n-dimensional pseudomanifold K such that $\dot{K}$ is neither empty nor an $(n - 1)$-dimensional pseudomanifold.

D SIMPLICIAL MAPS

In the first four exercises K will be a finite n-dimensional pseudomanifold, where $n > 0$, with nonempty boundary $\dot{K}$, K' will be a simplicial subdivision of K, and φ: $K' \to K$ will be a simplicial map such that $\varphi \mid \dot{K}'$ maps $\dot{K}'$ to $\dot{K}$ and is a simplicial approximation to the identity map $|\dot{K}'| \subset |\dot{K}|$. Furthermore, s^{n-1} will be a fixed $(n-1)$-simplex of $\dot{K}$ and s^n will be the unique n-simplex of K having s^{n-1} as a face.

1 For each n-simplex s' of K' let $\alpha(s')$ be the number of $(n-1)$-faces of s' mapped onto s^{n-1} by φ. Prove that $\alpha(s') = 1$ if and only if φ maps s' onto s^n and that $\alpha(s') = 0$ or 2 otherwise.

2 Prove that the number of n-simplexes of K' mapped onto s^n by φ has the same parity as the number of $(n-1)$-simplexes of $\dot{K}'$ mapped onto s^{n-1} by φ. [*Hint:* They both have the same parity as $\Sigma\, \alpha(s')$, the summation being over all n-simplexes s' of K'.]

3 *Sperner lemma.* Prove that the number of n-simplexes of K' mapped onto s^n by φ is odd. (*Hint:* Use induction on n.)

4 Prove that $|\dot{K}|$ is not a retract of $|K|$.

5 *Brouwer fixed-point theorem.* Prove that every continuous map of E^n to itself has a fixed point.

E SIMPLICIAL MAPPING CYLINDERS

Let φ: $K \to L$ be a simplicial map between simplicial complexes whose vertex sets are disjoint. We assume that the vertices of K are simply ordered. The *simplicial mapping cylinder* M of φ is the simplicial complex whose vertex set is the union of the vertex sets of K and L and whose simplexes are the simplexes of K and of L and all subsets of sets of the form $\{v_0, \ldots, v_k, \varphi(v_k), \ldots, \varphi(v_p)\}$, where $\{v_0, v_1, \ldots, v_p\}$ is a simplex of K and $v_0 < v_1 < \cdots < v_p$ in the simple ordering of the vertices of K.

1 Prove that the inclusion maps i: $K \subset M$ and j: $L \subset M$ are simplicial maps. If ρ: $M \to L$ is defined by $\rho(v) = \varphi(v)$ for v a vertex of K and $\rho(v') = v'$ for v' a vertex of L, then prove that ρ is a simplicial map such that $\varphi = \rho \circ i$ and $\rho \circ j = 1_L$.

2 If K is finite, prove that $j \circ \rho$ and 1_M are contiguous.

3 Prove that $|L|$ is a deformation retract of $|M|$.

F EDGE-PATH GROUPS

1 Prove that if K is a simplicial complex, there is a one-to-one correspondence between equivalence classes of local systems on $|K|$ with values in $\mathcal{C}$ and natural equivalence classes of covariant functors from the edge-path groupoid of K to $\mathcal{C}$.

2 *Van Kampen's theorem for simplicial complexes.*[1] Let K be a connected simplicial complex with connected subcomplexes L_1 and L_2 such that $L_1 \cap L_2$ is connected and $K = L_1 \cup L_2$. Let v_0 be a vertex of $L_1 \cap L_2$ and let i_1: $(L_1 \cap L_2, v_0) \subset (L_1, v_0)$ and i_2: $(L_1 \cap L_2, v_0) \subset (L_2, v_0)$. Prove that $E(K,v_0)$ is isomorphic to the quotient group of the free product of $E(L_1,v_0)$ with $E(L_2,v_0)$ by the normal subgroup generated by the set

$$\{(i_{1\#}[\zeta]) \circ (i_{2\#}[\zeta]^{-1}) \mid [\zeta] \in E(L_1 \cap L_2, v_0)\}$$

3 If G is a finitely presented group, prove that there is a finite connected two-dimensional simplicial complex K whose edge-path group is isomorphic to G.

[1] For the topological case see P. Olum, Non-abelian cohomology and Van Kampen's theorem, *Annals of Mathematics*, vol. 68, pp. 658–668, 1958.

4 Let X be a space with base point $x_0 \in X$. Prove that there exists a polyhedron Y, with base point $y_0 \in Y$, and a continuous map $f: (Y,y_0) \to (X,x_0)$ such that $f_\#: \pi(Y,y_0) \approx \pi(X,x_0)$.

G NERVES OF COVERINGS

If $\mathcal{U} = \{U\}$ in an open covering of a space X and $K(\mathcal{U})$ is its nerve, a *canonical map* $f: X \to |K(\mathcal{U})|$ is a continuous map such that $f^{-1}(\text{st } U) \subset U$ for every $U \in \mathcal{U}$.

1 If $\mathcal{U}$ is a locally finite open covering of X, prove that there is a one-to-one correspondence between canonical maps $X \to |K(\mathcal{U})|$ and partitions of unity subordinate to $\mathcal{U}$.

2 If $\mathcal{U}$ is a locally finite open covering of X, prove that any two canonical maps $X \to |K(\mathcal{U})|$ are homotopic.

If $\mathcal{U}$ and $\mathcal{V}$ are open coverings of X, with $\mathcal{V}$ a refinement of $\mathcal{U}$, a *canonical projection* from $\mathcal{V}$ to $\mathcal{U}$ is a function φ which assigns to each $V \in \mathcal{V}$ an element $\varphi(V) \in \mathcal{U}$ *such that* $V \subset \varphi(V)$.

3 Prove that a canonical projection from $\mathcal{V}$ to $\mathcal{U}$ defines a simplicial map $K(\mathcal{V}) \to K(\mathcal{U})$ and any two canonical projections from $\mathcal{V}$ to $\mathcal{U}$ define contiguous simplicial maps $K(\mathcal{V}) \to K(\mathcal{U})$.

4 If $\varphi: K(\mathcal{V}) \to K(\mathcal{U})$ is a canonical projection and $f: X \to |K(\mathcal{V})|$ is a canonical map, prove that the composite $|\varphi| \circ f: X \to |K(\mathcal{U})|$ is a canonical map.

5 Let X be a paracompact space and let $g: X \to |K|$ be a continuous map (where K is a simplicial complex). Prove that there exists a locally finite open covering $\mathcal{U}$ of X and a simplicial map $\varphi: K(\mathcal{U}) \to K$ such that for any canonical map $f: X \to |K(\mathcal{U})|$ the composite $|\varphi| \circ f$ is homotopic to g. [*Hint:* Choose $\mathcal{U}$ to be any locally finite open refinement of the open covering $\{g^{-1}(\text{st } v) \mid v \text{ a vertex of } K\}$, and for $U \in \mathcal{U}$ choose $\varphi(U)$ a vertex of K such that $U \subset g^{-1}(\text{st } \varphi(U))$.]

6 Let X be a compact Hausdorff space and let K be a simplicial complex. Prove that there is a bijection

$$\lim_{\rightarrow} \{[K(\mathcal{U});K]\} \approx [X;|K|]$$

where the direct limit is with respect to the family of finite open coverings of X directed by refinement with maps induced by canonical projections and the bijection is induced by canonical maps.

H DIMENSION THEORY

A topological space X is said to have *dimension* $\leq n$, abbreviated $\dim X \leq n$, if every open covering of X has an open refinement whose nerve is a simplicial complex of dimension $\leq n$. If $\dim X \leq n$ but $\dim X \not\leq n - 1$, then X is said to have *dimension* n, denoted by $\dim X = n$. If $\dim X \not\leq n$ for any n, we write $\dim X = \infty$.

1 If A is a closed subset of X, prove that $\dim A \leq \dim X$.

2 If K is a finite simplicial complex with $\dim K \leq n$, prove that $\dim |K| \leq n$.

3 If s is an n-simplex, prove that $\dim |s| = n$. (*Hint:* Let $\mathcal{U}$ be the open covering of $|s|$ of stars of the vertices of s and assume that there is a refinement $\mathcal{V}$ of $\mathcal{U}$ such that $\dim K(\mathcal{V}) \leq n - 1$. Let K' be a subdivision of s finer than $\mathcal{V}$. There are simplicial maps $K' \to K(\mathcal{V}) \to s$ whose composite λ is a simplicial approximation to the identity map $|K'| \subset |s|$.)

4 Let X be a paracompact space with $\dim X \leq n$. Prove that any map $X \to S^m$, with $m > n$, is null homotopic.

5 Let X be a compact metric space and let C be the space of maps $f: X \to \mathbf{R}^{2n+1}$ topologized by the metric

$$d(f,g) = \sup \{ \| f(x) - g(x) \| \mid x \in X \}$$

Prove that C is a complete metric space, and if

$$C_m = \{ f \in C \mid \operatorname{diam} f^{-1}(z) < \frac{1}{m} \text{ for all } z \in \mathbf{R}^{2n+1} \}$$

then show that C_m is an open subset of C for every positive integer m and $\bigcap C_m$ is the set of homeomorphisms of X into $\mathbf{R}^{2n+1}$.

6 If X is a compact metric space of dimension $\leq n$, prove that C_m is a dense subset of C for every positive integer m. [*Hint:* Let $\mathfrak{U}$ be a finite open covering of X by sets of diameter $< 1/m$ such that $\dim K(\mathfrak{U}) \leq n$ and let $h: |K(\mathfrak{U})| \to \mathbf{R}^{2n+1}$ be a realization of $K(\mathfrak{U})$. If $f: X \to |K(\mathfrak{U})|$ is any canonical map, then $h \circ f \in C_m$. Given $g: X \to \mathbf{R}^{2n+1}$ and given $\varepsilon > 0$, show that it is possible to choose $\mathfrak{U}$ and h as above, so that $d(h \circ f, g) < \varepsilon$.]

7 If X is a compact metric space of dimension $\leq n$, prove that X can be embedded in $\mathbf{R}^{2n+1}$ (in fact, the set of homeomorphisms of X into $\mathbf{R}^{2n+1}$ is dense in C).

CHAPTER FOUR

HOMOLOGY

THIS CHAPTER INTRODUCES THE CONCEPT OF HOMOLOGY THEORY, WHICH IS OF fundamental importance in algebraic topology. A homology theory involves a sequence of covariant functors H_n to the category of abelian groups, and we shall define homology theories on two categories–the singular homology theory on the category of topological pairs and the simplicial homology theory on the category of simplicial pairs. The former is topologically invariant by definition and is formally easier to work with, while the latter is easier to visualize geometrically and by definition is effectively computable for finite simplicial complexes. The two theories are related by the basic result that the singular homology of a polyhedron is isomorphic to the simplicial homology of any of its triangulating simplicial complexes.

The functor H_n measures the number of "n-dimensional holes" in the space (or simplicial complex), in the sense that the n-sphere S^n has exactly one n-dimensional hole and no m-dimensional holes if $m \neq n$. A 0-dimensional hole is a pair of points in different path components, and so H_0 measures path connectedness. The functors H_n measure higher dimensional connectedness, and some of the applications of homology are to prove higher dimensional analogues of results obtainable in low dimensions by using connectedness considerations.

Sections 4.1 and 4.2 are devoted to the definition of the category of chain complexes and to an appropriate concept of homotopy in this category. Homology theory is introduced as a sequence of covariant functors naturally defined from the category of chain complexes to the category of abelian groups.

Simplicial homology theory is defined by means of a covariant functor from the category of simplicial complexes to the category of chain complexes. We study it in detail in Sec. 4.3, where it is shown that two different definitions (one based on oriented simplexes, the other on ordered simplexes) are isomorphic. In similar fashion, singular homology theory is defined via a covariant functor from the category of topological spaces to the category of chain complexes. Its basic properties are considered in Sec. 4.4, where it is shown that "small" singular simplexes suffice to define singular homology.

Section 4.5 introduces the concept of exact sequence. All the homology functors H_n occur together in the exact sequences of homology, and it is for this reason that we consider all these functors simultaneously, rather than one at a time. Section 4.6 is devoted to the exact Mayer-Vietoris sequences connecting the homology of the union of two spaces (or simplicial complexes), the homology of the spaces, and the homology of their intersection. We use these to prove the isomorphism of the simplicial homology groups of a simplicial complex with the singular homology groups of its corresponding space.

Section 4.7 contains some applications of homology theory. We prove that euclidean spaces of different dimensions are not homeomorphic. We also prove the Brouwer fixed-point theorem and the more general Lefschetz fixed-point theorem. Finally, we prove Brouwer's generalization of the Jordan curve theorem (that an $(n - 1)$-sphere imbedded in S^n separates S^n into two components), and we establish the invariance of domain. Section 4.8 contains a discussion of the axiomatic characterization of homology given by Eilenberg and Steenrod, as well as some related concepts.

1 CHAIN COMPLEXES

This section introduces the category of chain complexes and chain maps and the homology functor on this category. We also define covariant functors from the category of simplicial complexes and from the category of topological spaces to the category of chain complexes. The composites of these and the homology functor define homology functors on the category of simplicial complexes and on the category of topological spaces.

A *differential group* C consists of an abelian group C and an endomorphism $\partial: C \to C$ such that $\partial\partial = 0$. The endomorphism ∂ is called the *differential*, or *boundary operator* of C. There is a category whose objects are differential groups and whose morphisms are homomorphisms commuting with the differentials.

For a differential group C there is a subgroup of *cycles* $Z(C) = \ker \partial$ and a subgroup of *boundaries* $B(C) = \operatorname{im} \partial$. Because $\partial\partial = 0$, $B(C) \subset Z(C)$. The *homology group* $H(C)$ is defined to be the quotient group

$$H(C) = Z(C)/B(C)$$

The elements of $H(C)$ are called *homology classes.* If z is a cycle, its homology class in $H(C)$ is denoted by $\{z\}$. Two cycles z_1 and z_2 are *homologous,* denoted by $z_1 \sim z_2$, if their difference is a boundary, that is, if $\{z_1\} = \{z_2\}$.

If $\tau\colon C \to C'$ is a homomorphism of differential groups commuting with the differentials, then τ maps cycles of C to cycles of C' and boundaries of C to boundaries of C'. Therefore τ induces a homomorphism

$$\tau_*\colon H(C) \to H(C')$$

such that $\tau_*\{z\} = \{\tau(z)\}$ for $z \in Z(C)$. Because $(\tau_1\tau_2)_* = \tau_{1*}\tau_{2*}$, there is a covariant functor from the category of differential groups to the category of groups which assigns to a differential group C its homology group $H(C)$ and to a homomorphism τ its induced homomorphism τ_*.

A *graded group* $C = \{C_q\}$ consists of a collection of abelian groups C_q indexed by the integers. Elements of C_q are said to have *degree* q. A *homomorphism* $\tau\colon C \to C'$ of *degree d* from one graded group to another consists of a collection $\tau = \{\tau_q\colon C_q \to C'_{q+d}\}$ of homomorphisms indexed by the integers. We shall omit the subscript in τ_q where there is no likelihood of confusion. It is obvious that the composite of homomorphisms of degrees d and d' is a homomorphism of degree $d + d'$, and that there thus is a category of graded groups and homomorphisms (with each homomorphism having some degree). It has a subcategory of graded groups and homomorphisms of fixed degree 0. Because the sum of two homomorphisms from C to C' of degree 0 is again a homomorphism from C to C' of degree 0, hom (C,C') is an abelian group [hom (C,C') being the set of morphisms in the category whose morphisms are homomorphisms of degree 0].

A *differential graded group* (sometimes abbreviated to DG group) is a graded group that has a differential compatible with the graded structure (that is, the differential is of degree r for some r). A *chain complex* is a differential graded group in which the differential is of degree -1. Thus a chain complex C consists of a sequence of abelian groups C_q and homomorphisms

$$\partial_q\colon C_q \to C_{q-1}$$

indexed by the integers such that the composite

$$C_{q+1} \xrightarrow{\partial_{q+1}} C_q \xrightarrow{\partial_q} C_{q-1}$$

is the trivial homomorphism. The elements of C_q are called *q-chains* of the complex. Most of the chain complexes we consider will have the additional property that $C_q = 0$ for $q < 0$. Such a complex is said to be *nonnegative.* A *free chain complex* is a chain complex in which C_q is a free abelian group for every q.

For a chain complex the group of cycles $Z(C)$ is a graded group consisting of the collection $\{Z_q(C) = \ker \partial_q\}$, and the group of boundaries $B(C)$ is a graded group consisting of $\{B_q(C) = \operatorname{im} \partial_{q+1}\}$. The homology group $H(C)$ is a graded group consisting of $\{H_q(C) = Z_q(C)/B_q(C)\}$.

A *chain map* $\tau: C \to C'$ (also called a *chain transformation*) between chain complexes is a homomorphism of degree 0 commuting with the differentials. Thus τ is a collection $\{\tau_q: C_q \to C'_q\}$ such that commutativity holds in each square

$$\begin{array}{ccc} C_q & \xrightarrow{\partial_q} & C_{q-1} \\ {\scriptstyle \tau_q}\downarrow & & \downarrow{\scriptstyle \tau_{q-1}} \\ C'_q & \xrightarrow{\partial'_q} & C'_{q-1} \end{array}$$

It is clear that there is a *category of chain complexes* whose objects are chain complexes and whose morphisms are chain maps. It is also clear that if C and C' are two objects in this category, hom (C,C') is an abelian group.

If $\tau: C \to C'$ is a chain map, its *induced homomorphism*

$$\tau_*: H(C) \to H(C')$$

is the homomorphism of degree 0 such that $(\tau_*)_q\{z\} = \{\tau_q(z)\}$ for $z \in Z_q(C)$. The following theorem is easily verified.

1 THEOREM *There is a covariant functor from the category of chain complexes to the category of graded groups and homomorphisms of degree* 0 *which assigns to a chain complex C its homology group H*(*C*) *and to a chain map τ its induced homomorphism τ_*. For any two chain complexes the map $\tau \to \tau_*$ is a homomorphism from* hom (C,C') *to* hom $(H(C),H(C'))$. ■

A *subcomplex* C' of a chain complex C, denoted by $C' \subset C$, is a chain complex such that $C'_q \subset C_q$ and $\partial'_q = \partial_q \mid C'_q$ for all q. There is then an inclusion map $i: C' \subset C$ consisting of the collection of inclusion maps $\{C'_q \subset C_q\}$. There is also a *quotient chain complex* $C/C' = \{C_q/C'_q\}$ with boundary operator induced from that of C by passing to the quotient. The collection of projections $\{C_q \to C_q/C'_q\}$ is the *projection chain map* $C \to C/C'$.

To describe a covariant functor from the category of simplicial complexes to the category of free chain complexes, let K be a simplicial complex. An *oriented q-simplex* of K is a q-simplex $s \in K$ together with an equivalence class of total orderings of the vertices of s, two orderings being equivalent if they differ by an even permutation of the vertices. If $v_0, v_1, \ldots, v_q$ are the vertices of s, then $[v_0,v_1, \ldots ,v_q]$ denotes the oriented q-simplex of K consisting of the simplex s together with the equivalence class of the ordering $v_0 < v_1 < \cdots < v_q$ of its vertices.

For $q < 0$ there are no oriented q-simplexes. For every vertex v of K there is a unique oriented 0-simplex $[v]$, and to every q-simplex, with $q \geq 1$, there correspond exactly two oriented q-simplexes. Let $C_q(K)$ be the abelian group generated by the oriented q-simplexes σ^q with the relations $\sigma_1{}^q + \sigma_2{}^q = 0$

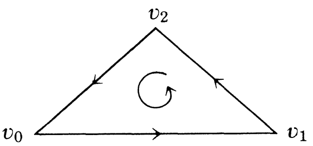
if $\sigma_1{}^q$ and $\sigma_2{}^q$ are different oriented q-simplexes corresponding to the same q-simplex of K. Then $C_q(K) = 0$ for $q < 0$, and for $q \geq 0$ $C_q(K)$ is a free abelian group with rank equal to the number of q-simplexes of K. If K is empty, $C_q(K) = 0$ for all q.

We define homomorphisms ∂_q: $C_q(K) \to C_{q-1}(K)$ for $q \geq 1$ by defining them on the generators by

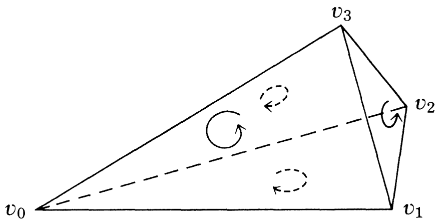
$$(a) \qquad \partial_q[v_0,v_1, \ldots ,v_q] = \sum_{0 \leq i \leq q} (-1)^i[v_0,v_1, \ldots ,\hat{v}_i, \ldots ,v_q]$$

where $[v_0,v_1, \ldots ,\hat{v}_i, \ldots ,v_q]$ denotes the oriented $(q-1)$-simplex obtained by omitting v_i. If $\sigma_1{}^q + \sigma_2{}^q = 0$ in $C_q(K)$, then it is easily verified that $\partial_q(\sigma_1{}^q) + \partial_q(\sigma_2{}^q) = 0$ in $C_{q-1}(K)$. Therefore ∂_q extends to a homomorphism from $C_q(K)$ to $C_{q-1}(K)$. For $q \leq 0$ we define ∂_q to be the trivial homomorphism from $C_q(K)$ to $C_{q-1}(K)$. It is not difficult to show that $\partial_q\partial_{q+1} = 0$ for all q. Therefore there is a free nonnegative chain complex $C(K) = \{C_q(K),\partial_q\}$, which is called the *oriented chain complex of* K. Its homology group, denoted by $H(K)$, is a graded group $\{H_q(K) = H_q(C(K))\}$, called the *oriented homology group of* K. $H_q(K)$ is called the *qth oriented homology group of* K.

If K is realized in some euclidean space, the oriented q-simplexes of K are q-simplexes of K together with orientations, in the sense of linear algebra, of the affine varieties spanned by them. The boundary of an oriented q-simplex is the sum of its oriented $(q-1)$-faces, with each face oriented by the orientation compatible with that of the q-simplex, as shown in the diagrams.



$\partial[v_0,v_1,v_2]$



$\partial[v_0,v_1,v_2,v_3]$

An oriented q-cycle z of K is a "closed" collection of oriented q-simplexes, with each $(q-1)$-simplex lying in the boundary of z the same number of times with each orientation. $H_q(K)$ is the group of equivalence classes of these q-cycles, two cycles being equivalent if their difference is a boundary. Thus $H_q(K)$ corresponds intuitively to the group generated by the q-dimensional "holes" in $|K|$.

It is convenient to add more generators and more relations to the chain groups $C_q(K)$. If v_0, v_1, . . . , v_q are vertices (not necessarily distinct) of some simplex of K, we define $[v_0,v_1, \ldots ,v_q] \in C_q(K)$ to be 0 if the vertices are not distinct and to be the oriented q-simplex as defined above if they are distinct. Then equation (a) remains correct for these added generators (that is, if the vertices $v_0,v_1, \ldots ,v_q$ are not all distinct, the left-hand side of equation (a) is 0 and the right-hand side can also be verified to be 0).

If $\varphi: K_1 \to K_2$ is a simplicial map, there is an associated chain map $C(\varphi): C(K_1) \to C(K_2)$ defined by

$$(b) \qquad C(\varphi)([v_0,v_1, \ldots ,v_q]) = [\varphi(v_0), \varphi(v_1), \ldots ,\varphi(v_q)]$$

Note that if $v_0, v_1, \ldots , v_q$ are distinct vertices of some simplex of K_1, then $\varphi(v_0), \varphi(v_1), \ldots , \varphi(v_q)$ are vertices of some simplex of K_2 but are not necessarily distinct. Therefore the right-hand side of equation (b) above would not be defined unless we had defined $[v_0,v_1, \ldots ,v_q]$ as an element of C_q, whether or not the terms v_i are distinct. It is easy to verify that $C(\varphi)$ is a chain map.

2 THEOREM *There is a covariant functor C from the category of simplicial complexes to the category of chain complexes which assigns to K its oriented chain complex C(K).* ■

The composite of the functor C and the homology functor is a covariant functor, called the *oriented homology functor,* from the category of simplicial complexes to the category of graded groups. To a simplicial complex K it assigns the graded group $H(K) = \{H_q(K) = H_q(C(K))\}$, and to a simplicial map $\varphi: K_1 \to K_2$ it assigns the homomorphism $\varphi_*: H(K_1) \to H(K_2)$ of degree 0 induced by $C(\varphi): C(K_1) \to C(K_2)$. If L is a subcomplex of K, and $i: L \subset K$, then $C(i): C(L) \to C(K)$ is a monomorphism by means of which we identify $C(L)$ with a subcomplex of $C(K)$.

We next describe the singular chain functor from the category of topological spaces to the category of chain complexes. Let $p_0, p_1, p_2, \ldots$ be an infinite sequence of different elements fixed once and for all. For $q \geq 0$ let Δ^q be the space of the simplicial complex consisting of all nonempty subsets of $\{p_0,p_1, \ldots ,p_q\}$ (therefore Δ^q is the closed simplex $|p_0,p_1, \ldots ,p_q|$). For $q \geq 0$ and $0 \leq i \leq q + 1$ let

$$e^i_{q+1}: \Delta^q \to \Delta^{q+1}$$

be the linear map defined by the vertex map

$$e^i_{q+1}(p_j) = \begin{cases} p_j & j < i \\ p_{j+1} & j \geq i \end{cases}$$

Then $e^i_{q+1}(\Delta^q)$ is the closed simplex $|p_0,p_1, \ldots ,\hat{p}_i, \ldots ,p_{q+1}|$ in Δ^{q+1} opposite the vertex p_i, and direct computation shows that

3 *If* $0 \leq j < i \leq q + 1$, *then* $e^i_{q+2}e^j_{q+1} = e^j_{q+2}e^{i-1}_{q+1}$. ■

Let X be a topological space. For $q \geq 0$ a *singular q-simplex* σ *of* X is defined to be a continuous map

$$\sigma: \Delta^q \to X$$

For $q > 0$ and $0 \leq i \leq q$ the *ith face of* σ, denoted by $\sigma^{(i)}$, is defined to be the singular $(q - 1)$-simplex of X which is the composite

$$\sigma^{(i)} = \sigma \circ e_q^i: \Delta^{q-1} \to \Delta^q \to X$$

It follows from statement 3 that

4 *If $q > 1$ and $0 \leq j < i \leq q$, then $(\sigma^{(i)})^{(j)} = (\sigma^{(j)})^{(i-1)}$.* ■

The *singular chain complex of* X, denoted by $\Delta(X)$, is defined to be the free nonnegative chain complex $\Delta(X) = \{\Delta_q(X),\partial_q\}$, where $\Delta_q(X)$ is the free abelian group generated by the singular q-simplexes of X for $q \geq 0$ [and $\Delta_q(X) = 0$ for $q < 0$], and for $q \geq 1$, ∂_q is defined by the equation

$$\partial_q(\sigma) = \sum_{0 \leq i \leq q} (-1)^i \sigma^{(i)}$$

This is a chain complex because $\partial_q \partial_{q+1} = 0$ is an immediate consequence of statement 4. If X is empty, $\Delta_q(X) = 0$ for all q.

If $f\colon X \to Y$ is continuous, there is a chain map

$$\Delta(f)\colon \Delta(X) \to \Delta(Y)$$

defined by $\Delta(f)(\sigma) = f \circ \sigma$ for a singular q-simplex $\sigma\colon \Delta^q \to X$. Then $\Delta(f)$ is a chain map, and we have the following result.

5 THEOREM *There is a covariant functor Δ from the category of topological spaces to the category of chain complexes which assigns to X its singular chain complex $\Delta(X)$.* ■

The composite of the functor Δ and the homology functor is a covariant functor, called the *singular homology functor*, from the category of topological spaces to the category of graded groups. To a space X it assigns the graded group $H(X) = \{H_q(X) = H_q(\Delta(X))\}$ and to a map $f\colon X \to Y$ it assigns the homomorphism

$$f_*\colon H(X) \to H(Y)$$

of degree 0 induced by $\Delta(f)\colon \Delta(X) \to \Delta(Y)$. $H_q(X)$ is called the *qth singular homology group of* X. If A is a subspace of X and $i\colon A \subset X$, then the map $\Delta(i)\colon \Delta(A) \to \Delta(X)$ is a monomorphism by means of which we identify $\Delta(A)$ with a subcomplex of $\Delta(X)$.

The category of chain complexes has arbitrary sums and products of indexed collections. That is, if $\{C^j\}_{j \in J}$ is an indexed collection of chain complexes, there is a *sum chain complex* $\bigoplus C^j$ and a *product chain complex* $\times\, C^j$ in which $(\bigoplus C^j)_q = \bigoplus C^j_q$ and $(\times\, C^j)_q = \times\, C^j_q$ for every q. It follows that $Z_q(\bigoplus C^j) = \bigoplus Z_q(C^j)$ and $B_q(\bigoplus C^j) = \bigoplus B_q(C^j)$ and $Z_q(\times\, C^j) = \times\, Z_q(C^j)$ and $B_q(\times\, C^j) = \times\, B_q(C^j)$ for all q. Therefore $H(\bigoplus C^j) = \bigoplus H(C^j)$ and $H(\times\, C^j) = \times\, H(C^j)$.

6 THEOREM *On the category of chain complexes the homology functor commutes with sums and with products.* ■

The category of chain complexes also has direct and inverse limits (whose qth chain groups are appropriate limits of the qth chain groups of the factors).

7 THEOREM *The homology functor commutes with direct limits.*

PROOF Let $\{C^\alpha,\tau_\alpha{}^\beta\}$ be a direct system of chain complexes and let $\{C,i_\alpha\}$ be the direct limit of this system (that is, i_α: $C^\alpha \to C$, and if $\alpha \le \beta$, then $i_\alpha = i_\beta \circ \tau_\alpha{}^\beta$: $C^\alpha \to C^\beta \to C$). Then $\{H(C^\alpha),\tau_{\alpha *}^\beta\}$ is a direct system of graded groups, and we show that $\{H(C),i_{\alpha *}\}$ is the direct limit of this system.

We show that 1.3*a* of the Introduction is satisfied. Let $\{z\} \in H_q(C)$. Then $z = i_\alpha c^\alpha$ for some $c^\alpha \in (C^\alpha)_q$. Since

$$0 = \partial_q z = \partial_q i_\alpha c^\alpha = i_\alpha \partial_q{}^\alpha c^\alpha$$

there is β with $\alpha \le \beta$ such that $\tau_\alpha{}^\beta \partial_q{}^\alpha c^\alpha = 0$. Then $\tau_\alpha{}^\beta c^\alpha$ is a cycle of $(C^\beta)_q$ and $i_\beta \tau_\alpha{}^\beta c^\alpha = i_\alpha c^\alpha = z$. Therefore $i_{\beta *}\{\tau_\alpha{}^\beta c^\alpha\} = \{z\}$.

We show that 1.3*b* of the Introduction is also satisfied. Because we are dealing with the direct limit of groups, it suffices to show that if $\{z^\alpha\} \in H_q(C^\alpha)$ is in the kernel of $i_{\alpha *}$, then there is γ with $\alpha \le \gamma$ such that $\{z^\alpha\}$ is in the kernel of $\tau_{\alpha *}^\gamma$. If $i_{\alpha *}\{z^\alpha\} = 0$, then $i_\alpha z^\alpha = \partial_{q+1} c$ for some $c \in C_{q+1}$. Because $c = i_\beta c^\beta$ for some β, we have $i_\alpha z^\alpha = i_\beta \partial_{q+1}^\beta c^\beta$. Choose γ' so that α, $\beta \le \gamma'$. Then $i_{\gamma'}(\tau_\alpha{}^{\gamma'} z^\alpha - \tau_\beta{}^{\gamma'} \partial_{q+1}^\beta c^\beta) = 0$. Therefore there is γ with $\gamma' \le \gamma$ such that $\tau_{\gamma'}{}^\gamma(\tau_\alpha{}^{\gamma'} z^\alpha - \tau_\beta{}^{\gamma'}\partial_{q+1}^\beta c^\beta) = 0$. Then $\tau_\alpha{}^\gamma z^\alpha = \partial_{q+1}^\gamma(\tau_\beta{}^\gamma c^\beta)$, so $\tau_{\alpha *}^\gamma\{z^\alpha\} = 0$. ∎

It is false that the homology functor commutes with inverse limits. We present an example to show this.

8 EXAMPLE For any integer $n \ge 1$ let C_n be the chain complex with $(C_n)_q = 0$ if $q \ne 0$ or 1 and $(C_n)_1 \xrightarrow{(\partial_n)_1} (C_n)_0$ equal to $\mathbf{Z} \to \mathbf{Z}$, where the homomorphism is multiplication by 2. For each n let τ^n: $C_{n+1} \to C_n$ be the chain map which is multiplication by 3 on each chain group, and for $n \le m$ define $\tau_n{}^m$: $C_m \to C_n$ to be the composite $\tau_n{}^m = \tau^n\tau^{n+1} \cdots \tau^{m-1}$. Then $\{C_n,\tau_n{}^m\}$ is an inverse system whose inverse limit is the trivial chain complex. Therefore $H_q(\lim_\leftarrow\{C_n,\tau_n{}^m\}) = 0$ for all q. However, $H_0(C_n) = \mathbf{Z}_2$ for all n and τ_{n*}^m: $H_0(C_m) \approx H_0(C_n)$ for all $n \le m$. Therefore $\lim_\leftarrow \{H_0(C_n), \tau_{n*}^m\} \approx \mathbf{Z}_2$.

2 CHAIN HOMOTOPY

This section deals with homotopy in the category of chain complexes. For free chain complexes we prove that contractibility is equivalent to triviality of all the homology groups. This leads to discussion of a method for constructing chain maps and homotopies by a general procedure known as the method of acyclic models. The section closes with a definition of mapping cone of a chain map and its relation to the chain map.

Let τ, τ': $C \to C'$ be chain maps. A *chain homotopy* D from τ to τ', denoted by D: $\tau \simeq \tau'$, is a homomorphism $D = \{D_q\}$ from C to C' of degree 1 such that for all q

$$\partial'_{q+1} D_q + D_{q-1}\partial_q = \tau_q - \tau'_q: C_q \to C'_q$$

If there is a chain homotopy from τ to τ', we say that τ is *chain homotopic* to τ' and write $\tau \simeq \tau'$. It is trivial that chain homotopy is an equivalence relation in the set of chain maps from C to C'. The corresponding set of equivalence classes is denoted by $[C;C']$, and if $\tau: C \to C'$ is a chain map, its equivalence class is denoted by $[\tau]$.

1 LEMMA *The composites of chain-homotopic chain maps are chain homotopic.*

PROOF Assume $D: \tau \simeq \tau'$, where $\tau, \tau': C \to C'$, and $\bar{D}: \bar{\tau} \simeq \bar{\tau}'$, where $\bar{\tau}, \bar{\tau}': C' \to C''$. Then

$$\bar{\tau}D + \bar{D}\tau': C \to C' \to C''$$

is of degree 1 and is a chain homotopy from $\bar{\tau}\tau$ to $\bar{\tau}'\tau'$. ∎

It follows that there is a category whose objects are chain complexes and whose morphisms are chain homotopy classes. A chain map $\tau: C \to C'$ is called a *chain equivalence* if $[\tau]$ is an equivalence in the homotopy category of chain complexes. If there is a chain equivalence from C to C', we say that C and C' are *chain equivalent.*

2 THEOREM *If $\tau, \tau': C \to C'$ are chain homotopic, then*

$$\tau_* = \tau'_*: H(C) \to H(C')$$

PROOF Assume $D: \tau \simeq \tau'$. For any $z \in Z_q(C)$

$$\partial'_{q+1}D_q(z) = \tau_q(z) - \tau'_q(z)$$

showing that $\tau_q(z) \sim \tau'_q(z)$ and $\tau_*\{z\} = \tau'_*\{z\}$. ∎

A *chain contraction* of a chain complex C is a homotopy from the identity chain map 1_C to the zero chain map 0_C of C to itself. If there is a chain contraction of C, C is said to be *chain contractible.* C is said to be *acyclic* if $H(C) = 0$ (that is, $H_q(C) = 0$ for all q).

3 COROLLARY *A contractible chain complex is acyclic.*

PROOF Assume that C is a chain complex such that $1_C \simeq 0_C$. By theorem 2, $(1_C)_* = (0_C)_*$. However, $(1_C)_* = 1_{H(C)}$ and $(0_C)_* = 0_{H(C)}$. Therefore $1_{H(C)} = 0_{H(C)}$, which can happen only if $H(C) = 0$. ∎

The converse of corollary 3 is false.

4 EXAMPLE Let C be the chain complex with $C_q = 0$ for $q \neq 0, 1, 2$ and with $C_2 \xrightarrow{\partial_2} C_1 \xrightarrow{\partial_1} C_0$ equal to $\mathbf{Z} \xrightarrow{\alpha} \mathbf{Z} \xrightarrow{\beta} \mathbf{Z}_2$, where $\alpha(n) = 2n$, $\beta(2m) = 0$, and $\beta(2m+1) = 1$. Then C is acyclic but not contractible. In fact, if $D: 1_C \simeq 0_C$ were a chain contraction of C, then the homomorphism β would have a right inverse $D_0: \mathbf{Z}_2 \to \mathbf{Z}$, but any homomorphism $\mathbf{Z}_2 \to \mathbf{Z}$ is trivial.

If C is assumed to be a free chain complex, there is a converse of corollary 3.

5 THEOREM *A free chain complex is acyclic if and only if it is contractible.*

PROOF We show that if C is an acyclic free chain complex, it is contractible. For each q the map ∂_q is an epimorphism of C_q to $B_{q-1}(C) = Z_{q-1}(C)$. Because C_{q-1} is free, so is $Z_{q-1}(C)$, and there is a homomorphism

$$s_{q-1}\colon Z_{q-1}(C) \to C_q$$

which is a right inverse of ∂_q. Then $1_{C_q} - s_{q-1}\partial_q$ maps C_q to $Z_q(C)$, and we define $\{D_q\}$ by

$$D_q = s_q(1_{C_q} - s_{q-1}\partial_q)\colon C_q \to C_{q+1}$$

Then

$$\partial_{q+1}D_q + D_{q-1}\partial_q = 1_{C_q} - s_{q-1}\partial_q + s_{q-1}(1_{C_{q-1}} - s_{q-2}\partial_{q-1})\partial_q = 1_{C_q}$$

which shows that $\{D_q\}$ is a chain contraction of C. ∎

The method of proof of theorem 5 is a standard one used to construct chain maps and homotopies from a free chain complex to an acyclic chain complex. We now extend it to obtain a general method of constructing chain maps and chain homotopies, called the *method of acyclic models*. Repeated application of this method will be made in subsequent discussions. We consider a special version of the method of acyclic models which suffices for our applications.[1]

A *category with models* consists of a category $\mathcal{C}$ and a set $\mathfrak{M}$ of objects of $\mathcal{C}$ called *models*. Given a covariant functor G from a category $\mathcal{C}$ with models $\mathfrak{M}$ to the category of abelian groups, a *basis* for G is an indexed collection $\{g_j \in G(M_j)\}_{j\in J}$, where $M_j \in \mathfrak{M}$ such that for any object X of $\mathcal{C}$ the indexed collection

$$\{G(f)(g_j)\}_{j\in J, f\in \hom(M_j,X)}$$

is a basis for $G(X)$. If G has a basis, it is called a *free functor* on $\mathcal{C}$ with models $\mathfrak{M}$. In this case, if $h \in \hom(X,Y)$, then $G(h)$ maps each basis element of $G(X)$ to some basis element of $G(Y)$. Hence G is the composite of the covariant functor which assigns to X the set $\{G(f)(g_j) \mid j \in J, f \in \hom(M_j,X)\}$ with the covariant functor of example 1.2.2, which assigns to every set the free abelian group generated by it.

Let G be a covariant functor from a category $\mathcal{C}$ with models $\mathfrak{M}$ to the category of chain complexes. G is said to be *free* if G_q is a free functor to the category of abelian groups.

6 EXAMPLE Let K be a simplicial complex and let $\mathcal{C}(K)$ be the category defined by the partially ordered set of subcomplexes of K (as in example 1.1.11). Let $\mathfrak{M}(K) = \{\bar{s} \mid s \in K\}$ be models for $\mathcal{C}(K)$. We show that the covariant functor C which assigns to each subcomplex of K its oriented chain complex is a free nonnegative functor on $\mathcal{C}(K)$ with models $\mathfrak{M}(K)$ to the category of

[1] A general treatment can be found in S. Eilenberg and S. Mac Lane, Acyclic models, *American Journal of Mathematics*, vol. 79, pp. 189–199 (1953).

chain complexes. For each model $\bar{s}$ of dimension q choose once and for all an oriented q-simplex $\sigma(s)$ which generates $C_q(\bar{s})$. Then the indexed collection $\{\sigma(s) \mid \dim s = q\}_{s \in K}$ is a basis for C_q. Hence C_q is free with models $\mathfrak{M}(K)$.

7 EXAMPLE Let $\mathcal{C}$ be the category of topological spaces with models $\mathfrak{M} = \{\Delta^q \mid q \geq 0\}$ and let Δ be the singular chain functor. Then Δ is free and nonnegative on $\mathcal{C}$ with models $\mathfrak{M}$. In fact, if $\xi_q\colon \Delta^q \subset \Delta^q$, then the singleton $\{\xi_q \in \Delta_q(\Delta^q)\}$ is a basis for Δ_q.

Let G be a covariant functor on a category $\mathcal{C}$ to the category of chain complexes. Then there are covariant functors $H_q(G)$, for all q, from $\mathcal{C}$ to the category of abelian groups that assign to an object X the group $H_q(G(X))$. If $\mathcal{C}$ is a category with models $\mathfrak{M}$, a functor G from $\mathcal{C}$ to the category of chain complexes is said to be *acyclic in positive dimensions* if $H_q(G(M)) = 0$ for $q > 0$ and $M \in \mathfrak{M}$. We now establish the main result dealing with the construction of chain maps and homotopies.

8 THEOREM *Let $\mathcal{C}$ be a category with models $\mathfrak{M}$ and let G and G' be covariant functors from $\mathcal{C}$ to the category of chain complexes such that G is free and nonnegative and G' is acyclic in positive dimensions. Then*

(a) Any natural transformation $H_0(G) \to H_0(G')$ is induced by a natural chain map $\tau\colon G \to G'$.
(b) Two natural chain maps $\tau, \tau'\colon G \to G'$ inducing the same natural transformation $H_0(G) \to H_0(G')$ are naturally chain homotopic.

PROOF For every object X of $\mathcal{C}$ we must define a chain map $\tau(X)\colon G(X) \to G'(X)$ [or a chain homotopy $D(X)\colon \tau(X) \simeq \tau'(X)$] such that if $h\colon X \to Y$ is a morphism in $\mathcal{C}$, then

$$\tau(Y)G(h) = G'(h)\tau(X) \qquad [\text{or } D(Y)G(h) = G'(h)D(X)]$$

For $q \geq 0$ let $\{g_j \in G_q(M_j)\}_{j \in J_q}$ be a basis for G_q, where $M_j \in \mathfrak{M}$ for each $j \in J_q$. Then $G_q(X)$ has the basis

$$\{G_q(f)(g_j)\}_{j \in J_q, f \in \hom(M_j, X)}$$

It follows that $\tau_q(X)$ [or $D_q(X)$] is determined by the collection $\{\tau_q(M_j)(g_j)\}_{j \in J_q}$ and the equation

$$\tau_q(X)(\Sigma n_{ij} G_q(f_{ij})(g_j)) = \Sigma n_{ij} G'_q(f_{ij}) \tau_q(M_j)(g_j) \tag{a}$$

or by the collection $\{D_q(M_j) g_j\}_{j \in J_q}$ and the equation

$$D_q(X)(\Sigma n_{ij} G_q(f_{ij})(g_j)) = \Sigma n_{ij} G'_{q+1}(f_{ij}) D_q(M_j)(g_j) \tag{b}$$

We shall define $\tau_q(X)$ by induction on q so that

$$\partial \tau_q(X) = \tau_{q-1}(X) \partial \tag{c}$$

and define $D_q(X)$ by induction on q so that

$$\partial D_q(X) = \tau_q(X) - \tau'_q(X) - D_{q-1}(X) \partial \tag{d}$$

Having defined τ_i [or D_i] for $i < q$, where $q > 0$, it suffices to define $\tau_q(M_j)(g_j)$ for $j \in J_q$ so that

$$(e) \qquad \partial\tau_q(M_j)(g_j) = \tau_{q-1}(M_j)(\partial g_j)$$

and to define $D_q(M_j)(g_j)$ for $j \in J_q$ so that

$$(f) \qquad \partial D_q(M_j)(g_j) = \tau_q(M_j)(g_j) - \tau'_q(M_j)(g_j) - D_{q-1}(M_j)(\partial g_j)$$

since $\tau_q(X)$ [and $D_q(X)$] are then determined by equation (a) [or by (b)]. It will then be true that $\tau_q(X)$ [and $D_q(X)$] are natural and will satisfy equation (c) [and (d)].

Given a natural transformation $\varphi\colon H_0(G) \to H_0(G')$, the inductive definition of τ proceeds as follows. For $q = 0$ we define $\tau_0(M_j)(g_j)$ for $j \in J_0$ to be any element of $G'_0(M_j)$ such that $\{\tau_0(M_j)(g_j)\} = \varphi(M_j)\{g_j\}$. We use equation (a) to define $\tau_0(X)$ for all X. Then, for $g \in G_0(X)$, $\{\tau_0(X)(g)\} = \varphi(X)\{g\}$. In particular, for $j \in J_1$, $\tau_0(M_j)(\partial g_j)$ is a boundary in $G'_0(M_j)$. Hence we can define $\tau_1(M_j)(g_j) \in G'_1(M_j)$ so that $\partial\tau_1(M_j)(g_j) = \tau_0(M_j)(\partial g_j)$. We then use equation (a) to define $\tau_1(X)$ for all X. Assuming τ_i defined for $i < q$, where $q > 1$, so that equation (c) is satisfied, we observe that the right-hand side of equation (e) is a cycle of $G'_{q-1}(M_j)$. Because $q > 1$, $H_{q-1}(G'(M_j)) = 0$, and we define $\tau_q(M_j)(g_j)$ to satisfy equation (e). We next define $\tau_q(X)$ for all X to satisfy equation (a). This completes the definition of τ.

Given τ, $\tau'\colon G \to G'$ such that τ and τ' induce the same natural transformation $H_0(G) \to H_0(G')$, we define $D_0(M_j)(g_j)$ for $j \in J_0$ to be any element of $G'_1(M_j)$ whose boundary equals $\tau_0(M_j)(g_j) - \tau'_0(M_j)(g_j)$. Then $D_0(X)$ is defined for all X by equation (b). Assuming D_i defined for $i < q$, where $q > 0$, so that equation (d) is satisfied, we observe that the right-hand side of equation (f) is a cycle of $G'_q(M_j)$. Because $q > 0$, $H_q(G'(M_j)) = 0$, and this cycle is a boundary. We define $D_q(M_j)(g_j) \in G'_{q+1}(M_j)$ to satisfy equation (f), use equation (b) to define $D_q(X)$ for all X, and complete the definition of D. ∎

The last result provides another proof of theorem 5 for nonnegative complexes. In fact, if C is a free nonnegative chain complex, let $\mathcal{C}$ be the category consisting of one object X and one morphism 1_X and let C be regarded as a covariant functor on $\mathcal{C}$ with model $\{X\}$. Then C is a free nonnegative functor, and if C is an acyclic chain complex, the functor C is acyclic in positive dimensions. In this case, because 1_C and 0_C are chain transformations of C inducing the same homomorphism of $H_0(C) = 0$, it follows from theorem 8 that $1_C \simeq 0_C$, and C is contractible.

There is a useful algebraic object (related to the mapping cylinder of Sec. 1.4) which we now describe. Let $\tau\colon C \to C'$ be a chain map. The *mapping cone* of τ is the chain complex $\bar{C} = \{\bar{C}_q, \bar{\partial}_q\}$ defined by $\bar{C}_q = C_{q-1} \oplus C'_q$ and

$$\bar{\partial}_q(c,c') = (-\partial_{q-1}(c), \tau(c) + \partial'_q(c')) \qquad c \in C_{q-1},\ c' \in C'_q$$

The following result is trivial to verify.

9 LEMMA *$\bar{C}$ is a chain complex, and if C and C' are free chain complexes, so is $\bar{C}$.* ∎

The next theorem is the main reason for introducing mapping cones.

10 THEOREM *A chain map is a chain equivalence if and only if its mapping cone is chain contractible.*

PROOF Assume that $\tau: C \to C'$ is a chain equivalence. There exist $\tau': C' \to C$ and $D: C \to C$ and $D': C' \to C'$ such that $D: \tau'\tau \simeq 1_C$ and $D': \tau\tau' \simeq 1_{C'}$. Define $\bar{D}: \bar{C} \to \bar{C}$ by $\bar{D}(c,c') = (c_1,c_2)$, where

$$c_1 = D(c) + \tau'D'\tau(c) - \tau'\tau D(c) + \tau'(c')$$
$$c_2 = D'\tau D(c) - D'D'\tau(c) - D'(c')$$

A straightforward computation shows that $\bar{D}$ is a chain contraction of $\bar{C}$.

Conversely, assume that $\bar{D}$ is a chain contraction of $\bar{C}$. Define $\tau': C' \to C$ and $D: C \to C$ and $D': C' \to C'$ by the equations

$$(\tau'(c'), -D'(c')) = \bar{D}(0,c')$$
$$(D(c), \cdot) = \bar{D}(c,0)$$

Direct verification shows τ' to be a chain map and $D: \tau'\tau \simeq 1_C$ and $D': \tau\tau' \simeq 1_{C'}$, so τ is a chain equivalence. ∎

Combining this with theorem 5 and lemma 9 yields the following result.

11 COROLLARY *A chain map between free chain complexes is a chain equivalence if and only if its mapping cone is acyclic.* ∎

3 THE HOMOLOGY OF SIMPLICIAL COMPLEXES

This section begins with a discussion of augmented chain complexes and their reduced homology groups. Next we define the ordered chain complex of a simplicial complex and prove that it is chain equivalent to the oriented chain complex. We use this result to show that simplicial maps in the same contiguity class induce chain-homotopic chain maps. We also compute $H_0(K)$ in terms of the components of K. At the end of the section the relative homology groups and the Euler characteristic of a simplicial pair are defined.

In the category of nonempty simplicial complexes any simplicial complex P consisting of a single vertex is a terminal object. If K is a nonempty simplicial complex, the simplicial map $K \to P$ has a right inverse. Therefore the induced homology map $H(K) \to H(P)$ has a right inverse. Because $H_q(P) = 0$ if $q \neq 0$ and $H_0(P) \approx \mathbf{Z}$, it follows that there is an epimorphism $H_0(K) \to \mathbf{Z}$. Since $H_0(K) = C_0(K)/\partial_1 C_1(K)$, there is an epimorphism $\varepsilon: C_0(K) \to \mathbf{Z}$ such that $\varepsilon\partial_1 = 0$. Similarly, in the category of nonempty topological spaces X any one-point space is a terminal object. The same kind of considerations yield an epimorphism $\varepsilon: \Delta_0(X) \to \mathbf{Z}$ such that $\varepsilon\partial_1 = 0$. This motivates the following definition of augmentation.

An *augmentation* (*over* $\mathbf{Z}$) of a chain complex C is an epimorphism $\varepsilon: C_0 \to \mathbf{Z}$ such that $\varepsilon\partial_1: C_1 \to C_0 \to \mathbf{Z}$ is trivial. An *augmented chain complex* is a nonnegative chain complex C with augmentation. An augmentation ε of a chain complex can be regarded as an epimorphic chain map of C to the chain complex (also denoted by $\mathbf{Z}$) whose only nontrivial chain group is $\mathbf{Z}$ in degree 0. For this chain complex $\mathbf{Z}$, it is clear that $H_q(\mathbf{Z}) = 0$ for $q \neq 0$ and that $H_0(\mathbf{Z}) = \mathbf{Z}$. Therefore ε induces an epimorphism $\varepsilon_*: H_0(C) \to \mathbf{Z}$. Hence an augmented chain complex has a nontrivial homology group in degree 0.

The oriented chain complex $C(K)$ of a nonempty simplicial complex K is augmented by the homomorphism $\varepsilon: C_0(K) \to \mathbf{Z}$ defined by $\varepsilon([v]) = 1$ for every vertex v of K. The singular chain complex $\Delta(X)$ of a nonempty space X is augmented by the homomorphism $\varepsilon: \Delta_0(X) \to \mathbf{Z}$ defined by $\varepsilon(\sigma) = 1$ for every singular 0-simplex of X.

A chain map $\tau: C \to C'$ between augmented chain complexes *preserves augmentation* if $\varepsilon' \circ \tau = \varepsilon: C_0 \to \mathbf{Z}$. Note that τ preserves augmentation if and only if τ_* does—that is, if and only if $\varepsilon'_* \circ \tau_* = \varepsilon_*: H_0(C) \to \mathbf{Z}$. There is a category of augmented chain complexes and chain maps preserving augmentation. A chain homotopy in this category is any chain homotopy between chain maps in the category.

We want to divide out the functorial nontrivial part of $H_0(C)$ of an augmented chain complex C. The *reduced chain complex* $\tilde{C}$ *of* an augmented chain complex C is defined to be the chain complex defined by $\tilde{C}_q = C_q$ if $q \neq 0$, $\tilde{C}_0 = \ker \varepsilon$, and $\tilde{\partial}_q = \partial_q$ [note that $\partial_1(\tilde{C}_1) \subset \tilde{C}_0$ because $\varepsilon\partial_1 = 0$]. Thus $\tilde{C}$ is the kernel of the chain map $\varepsilon: C \to \mathbf{Z}$. If $\tau: C \to C'$ is a chain map preserving augmentation, τ induces a chain map $\tilde{C} \to \tilde{C}'$ between their reduced chain complexes. The homology group $H(\tilde{C})$ is called the *reduced homology group* of C and is denoted by $\tilde{H}(C)$. For a nonempty simplicial complex K we define $\tilde{H}(K) = \tilde{H}(C(K))$, and for a nonempty topological space X we define $\tilde{H}(X) = \tilde{H}(\Delta(X))$. Because the chain complex of an empty simplicial complex or an empty topological space has no augmentation, the reduced groups are not defined in this case. For that reason some of the arguments, which otherwise involve the reduced groups, require a special remark in the case of empty complexes or spaces.

Clearly, there is an inclusion chain map $\tilde{C} \subset C$.

1 LEMMA *If C is an augmented chain complex, then*

$$H_q(C) \approx \begin{cases} \tilde{H}_q(C) & q \neq 0 \\ \tilde{H}_0(C) \oplus \mathbf{Z} & q = 0 \end{cases}$$

PROOF Because $\mathbf{Z}$ is a free group, $C_0 \approx \tilde{C}_0 \oplus \mathbf{Z}$. Then $Z_q(C) = Z_q(\tilde{C})$ if $q \neq 0$, $Z_0(C) \approx Z_0(\tilde{C}) \oplus \mathbf{Z}$, and $B_q(C) = B_q(\tilde{C})$ for all q. ∎

It is clear that if $\tau: C \to C'$ is an augmentation-preserving chain map, the isomorphism of lemma 1 commutes with τ_*. It is also obvious that if C is a free augmented chain complex, $\tilde{C}$ is a free chain complex.

It follows from lemma 1 that if C is an augmented chain complex, $H_0(C) \neq 0$. Hence an augmented chain complex is never acyclic. The most that can be hoped for is that $\tilde{C}$ will be acyclic.

2 LEMMA *If C is an augmented chain complex, $\tilde{C}$ is chain contractible if and only if the augmentation ε is a chain equivalence of C with the chain complex* $\mathbf{Z}$.

PROOF Let $\bar{C}$ be the mapping cone of the chain map ε: $C \to \mathbf{Z}$. Then $\bar{C}_0 = \mathbf{Z}$ and $\bar{C}_q = C_{q-1}$ if $q > 0$, and $\bar{\partial}_1 = \varepsilon$ and $\bar{\partial}_q = -\partial_{q-1}$ for $q > 1$. By theorem 4.2.10, ε is a chain equivalence if and only if $\bar{C}$ is chain contractible.

We show that $\bar{C}$ is chain contractible if and only if $\tilde{C}$ is chain contractible. If $\bar{D}$: $\bar{C} \to \bar{C}$ is a chain contraction of $\bar{C}$, define $\tilde{D}$: $\tilde{C} \to \tilde{C}$ by $\tilde{D}_{q-1} = -\bar{D}_q \mid \tilde{C}_{q-1}$. Then $\tilde{D}$ is a chain contraction of $\tilde{C}$. Conversely, if $\tilde{D}$ is a chain contraction of $\tilde{C}$, define $\bar{D}$: $\bar{C} \to \bar{C}$ so that $\bar{D}_0$: $\mathbf{Z} \to C_0$ is a right inverse of ε: $C_0 \to \mathbf{Z}$, $\bar{D}_1$: $C_0 \to C_1$ is 0 on $\bar{D}_0(\mathbf{Z})$ and equal to $-\tilde{D}_0$ on $\tilde{C}_0$, and for $q > 1$, $\bar{D}_q$: $C_{q-1} \to C_q$ is equal to $-\tilde{D}_{q-1}$. Then $\bar{D}$ is a chain contraction of $\bar{C}$. ∎

Let $\mathcal{C}$ be a category with models $\mathfrak{M}$. A functor G' from $\mathcal{C}$ to the category of augmented chain complexes (and chain maps preserving augmentation) is said to be *acyclic* if $\tilde{G}'(M)$ is acyclic for $M \in \mathfrak{M}$. For augmented chain complexes there is the following form of the acyclic-model theorem.

3 THEOREM *Let $\mathcal{C}$ be a category with models $\mathfrak{M}$ and let G and G' be covariant functors from $\mathcal{C}$ to the category of augmented chain complexes such that G is free and G' is acyclic. There exist natural chain maps preserving augmentation from G to G', and any two are naturally chain homotopic.*

PROOF Let $\{g_j \in G_0(M_j)\}_{j \in J_0}$ be a basis for G_0. By lemma 1, ε': $H_0(G'(M_j)) \approx \mathbf{Z}$, and there is a unique $z_j \in H_0(G'(M_j))$ such that $\varepsilon'(z_j) = \varepsilon(g_j)$. A natural transformation $H_0(G) \to H_0(G')$ is defined by sending $\{\Sigma n_{ij} G_0(f_{ij})(g_j)\} \in H_0(G(X))$ to $\Sigma n_{ij} G_0'(f_{ij}) z_j \in H_0(G'(X))$ for $j \in J_0$ and $f_{ij} \in \hom(M_j, X)$ (where X is any object of $\mathcal{C}$), and this is the unique natural transformation $H_0(G) \to H_0(G')$ commuting with augmentation. The theorem now follows from theorem 4.2.8. ∎

With the hypotheses of theorem 3 there is a unique natural transformation from $H(G)$ to $H(G')$ commuting with augmentation. It is the homomorphism induced by any natural chain map preserving augmentation from G to G'.

4 COROLLARY *Let G and G' be free and acyclic covariant functors from a category $\mathcal{C}$ with models $\mathfrak{M}$ to the category of augmented chain complexes. Then G and G' are naturally chain equivalent; in fact, any natural chain map preserving augmentation from G to G' is a natural chain equivalence.*

PROOF Let τ: $G \to G'$ be a natural chain map preserving augmentation (which exists, by theorem 3). Also by theorem 3, there is a natural chain map τ': $G' \to G$ preserving augmentation and there are natural chain homotopies D: $\tau' \circ \tau \simeq 1_G$ and D': $\tau \circ \tau' \simeq 1_{G'}$. ∎

We are ultimately interested in comparing the chain complex $C(K)$ of a simplicial complex K with the singular chain complex $\Delta(|K|)$ of the space of K.

For this purpose we introduce a chain complex $\Delta(K)$ intermediate between them. Let K be a simplicial complex. An *ordered q-simplex* of K is a sequence $v_0, v_1, \ldots, v_q$ of $q + 1$ vertices of K which belong to some simplex of K. We use $(v_0,v_1, \ldots ,v_q)$ to denote the ordered q-simplex consisting of the sequence $v_0, v_1, \ldots, v_q$ of vertices. For $q < 0$ there are no ordered q-simplexes. An ordered 0-simplex (v) is the same as the oriented 0-simplex $[v]$. An ordered 1-simplex (v,v') is the same as an edge of K.

We define a free nonnegative chain complex, called the *ordered chain complex of* K, by $\Delta(K) = \{\Delta_q(K),\partial_q\}$, where $\Delta_q(K)$ is the free abelian group generated by the ordered q-simplexes of K [and $\Delta_q(K) = 0$ if $q < 0$] and ∂_q is defined by the equation

$$\partial_q(v_0,v_1, \ldots ,v_q) = \sum_{0\le i\le q} (-1)^i(v_0, \ldots ,\hat{v}_i, \ldots ,v_q)$$

Then $\Delta(K)$ is a chain complex, and if K is nonempty, $\Delta(K)$ is augmented by the augmentation $\varepsilon(v) = 1$ for any vertex v of K. If $\varphi\colon K_1 \to K_2$ is a simplicial map, there is an augmentation-preserving chain map

$$\Delta(\varphi)\colon \Delta(K_1) \to \Delta(K_2)$$

such that $\Delta(\varphi)(v_0,v_1, \ldots ,v_q) = (\varphi(v_0), \varphi(v_1), \ldots ,\varphi(v_q))$. Therefore we have the following theorem.

5 THEOREM *There is a covariant functor Δ from the category of nonempty simplicial complexes to the category of free augmented chain complexes which assigns to K the ordered chain complex $\Delta(K)$.* ∎

If L is a subcomplex of K and $i\colon L \subset K$, then $\Delta(i)\colon \Delta(L) \to \Delta(K)$ is a monomorphism by means of which we identify $\Delta(L)$ with a subcomplex of $\Delta(K)$. If $\mathcal{C}(K)$ is the category defined by the partially ordered set of subcomplexes of K and $\mathfrak{M}(K) = \{\bar{s} \mid s \in K\}$, then Δ is a free functor on $\mathcal{C}(K)$ with models $\mathfrak{M}(K)$.

For any simplicial complex K there is a surjective chain map (preserving augmentation if K is nonempty)

$$\mu\colon \Delta(K) \to C(K)$$

such that $\mu(v_0,v_1, \ldots ,v_q) = [v_0,v_1 \ldots ,v_q]$. Then μ is a natural transformation from Δ to C on the category of simplicial complexes. We shall show that it is a chain equivalence for every simplicial complex. The following theorem will be used to show that Δ and C are acyclic functors on $\mathcal{C}(K)$ with models $\mathfrak{M}(K)$.

6 THEOREM *Let K be a simplicial complex and let w be the simplicial complex consisting of a single vertex. Then $\tilde{\Delta}(K * w)$ and $\tilde{C}(K * w)$ are chain contractible.*

PROOF Since the proofs are analogous, we give the details only in the ordered complex. According to lemma 2, it suffices to prove that $\varepsilon\colon \Delta(K * w) \to \mathbf{Z}$ is a chain equivalence. Define a homomorphism $\tau: \mathbf{Z} \to \Delta_0(K * w)$ by $\tau(1) = (w)$ and regard it as a chain map $\tau: \mathbf{Z} \to \Delta(K * w)$. Then $\varepsilon \circ \tau = 1_{\mathbf{Z}}$. To show that $1_{\Delta(K*w)} \simeq \tau \circ \varepsilon$, define a chain homotopy $D: 1_{\Delta(K*w)} \simeq \tau \circ \varepsilon$ by the equation

$$D(v_0,v_1, \ldots ,v_q) = (w,v_0,v_1, \ldots ,v_q) \quad ■$$

Because a q-simplex is the join of a $(q - 1)$-face with the opposite vertex, we have the next result.

7 COROLLARY *For any simplex $s \in K$, $\tilde{\Delta}(\bar{s})$ and $\tilde{C}(\bar{s})$ are acyclic.* ■

8 THEOREM *For any simplicial complex K the natural chain map $\mu: \Delta(K) \to C(K)$ is a chain equivalence.*

PROOF If K is empty, $\Delta(K) = C(K)$ and μ is the identity, so the result is true in this case. If K is nonempty, it follows from corollary 7 that Δ and C are free acyclic functors on $\mathcal{C}(K)$ with models $\mathfrak{M}(K) = \{\bar{s} \mid s \in K\}$. By corollary 4, μ is a natural chain equivalence of Δ with C on $\mathcal{C}(K)$. In particular, $\mu: \Delta(K) \to C(K)$ is a chain equivalence. ■

The next result is that the functors Δ and C convert contiguity of simplicial maps into chain homotopy of chain maps. This result could also be proved by the method of acyclic models.

9 THEOREM *Let $\varphi, \varphi': K_1 \to K_2$ be in the same contiguity class. Then $\Delta(\varphi), \Delta(\varphi'): \Delta(K_1) \to \Delta(K_2)$ are chain homotopic, and in similar fashion $C(\varphi), C(\varphi'): C(K_1) \to C(K_2)$ are chain homotopic.*

PROOF Because chain homotopy is an equivalence relation, it suffices to prove the theorem for the case that φ and φ' are contiguous. An explicit chain homotopy $D: \Delta(\varphi) \simeq \Delta(\varphi')$ is defined by the formula

$$D(v_0,v_1, \ldots ,v_q) = \sum_{0 \le i \le q} (-1)^i(\varphi'(v_0), \ldots ,\varphi'(v_i), \varphi(v_i), \ldots ,\varphi(v_q))$$

That $C(\varphi)$ and $C(\varphi')$ are chain homotopic follows from the fact that $\Delta(\varphi)$ and $\Delta(\varphi')$ are chain homotopic and from theorem 8. ■

10 THEOREM *The homology groups of a complex are the direct sums of the homology groups of its components.*

PROOF If $\{K_j\}$ are the components of K, then $\bigoplus C(K_j) = C(K)$. The result follows from theorem 4.1.6. ■

If $\{K_\alpha\}$ is the collection of finite subcomplexes of K directed by inclusion, then $C(K) \approx \lim_{\to} \{C(K_\alpha)\}$. From theorem 4.1.7 we have the next result.

11 THEOREM *The homology groups of a simplicial complex are isomorphic to the direct limit of the homology groups of its finite subcomplexes.* ■

We are now ready to compute $H_0(K)$.

12 LEMMA *If K is a nonempty connected simplicial complex, then $\tilde{H}_0(K) = 0$.*

PROOF Let v_0 be a fixed vertex of K. For any vertex v of K there is an edge path $e_1e_2 \cdots e_r$ of K with origin at v_0 and end at v. Then $e_1 + e_2 + \cdots + e_r$ is a 1-chain $c_v \in \Delta_1(K)$ such that $\partial c_v = v - v_0$. Since $\varepsilon(\Sigma n_v v) = \Sigma n_v$, we see that if $\Sigma n_v v$ is any 0-chain of $\tilde{\Delta}_0(K)$, then $\Sigma n_v = 0$ and

$$\partial(\Sigma n_v c_v) = \Sigma n_v v - \Sigma n_v v_0 = \Sigma n_v v$$

Therefore $\tilde{H}_0(\Delta(K)) = 0$, and by theorem 8, $\tilde{H}_0(K) = 0$. ∎

13 COROLLARY *For any simplicial complex K, $H_0(K)$ is a free group whose rank equals the number of nonempty components of K.*

PROOF If K is empty, $H_0(K) = 0$, and the result is valid in this case. If K is nonempty and connected, it follows from lemmas 12 and 1 that $H_0(K) \approx \mathbf{Z}$. The general result then follows from theorem 10. ∎

If L is a subcomplex of K, there is a *relative oriented homology group* $H(K,L) = \{H_q(K,L) = H_q(C(K)/C(L))\}$ *of K modulo L*. If L is empty, $H(K,\varnothing) = H(K)$ is called the *absolute oriented homology group of K*. Similarly, there is a *relative ordered homology group $H(\Delta(K)/\Delta(L))$ of K modulo L* that generalizes the *absolute ordered homology group* $H(\Delta(K),\Delta(\varnothing))$. The relative homology groups $H(K,L)$ and $H(\Delta(K),\Delta(L))$ are covariant functors from the category of simplicial pairs to the category of graded groups.

If $H_q(K,L)$ is finitely generated (which will necessarily be true if $K - L$ contains only finitely many simplexes), it follows from the structure theorem (theorem 4.14 in the Introduction) that $H_q(K,L)$ is the direct sum of a free group and a finite number of finite cyclic groups $\mathbf{Z}_{n_1} \oplus \mathbf{Z}_{n_2} \oplus \cdots \oplus \mathbf{Z}_{n_k}$, where n_i divides n_{i+1} for $i = 1, \ldots, k-1$. The rank $\rho(H_q(K,L))$ is called the *qth Betti number of* (K,L), and the numbers $n_1, n_2, \ldots, n_k$ are called the *qth torsion coefficients of* (K,L). The qth Betti number and the qth torsion coefficients characterize $H_q(K,L)$ up to isomorphism.

A graded group C is said to be *finitely generated* if C_q is finitely generated for all q and $C_q = 0$ except for a finite set of integers q. It is obvious that if C is a finitely generated chain complex, $H(C)$ is a finitely generated graded group. Given a finitely generated graded group C, its *Euler characteristic* (also called the *Euler-Poincaré characteristic*), denoted by $\chi(C)$, is defined by

$$\chi(C) = \Sigma(-1)^q \rho(C_q)$$

14 THEOREM *Let C be a finitely generated chain complex. Then*

$$\chi(C) = \chi(H(C))$$

PROOF By definition, $Z_q(C) \subset C_q$ and the quotient group $C_q/Z_q(C) \approx B_{q-1}(C)$. By theorem 4.12 in the Introduction,

$$\rho(C_q) = \rho(Z_q(C)) + \rho(B_{q-1}(C))$$

Similarly, $H_q(C) = Z_q(C)/B_q(C)$, and again by theorem 4.12 of the Introduction,

$$\rho(Z_q(C)) = \rho(H_q(C)) + \rho(B_q(C))$$

Eliminating $\rho(Z_q(C))$, we have

$$\rho(C_q) = \rho(H_q(C)) + \rho(B_q(C)) + \rho(B_{q-1}(C))$$

Multiplying this equation by $(-1)^q$ and summing the resulting equations over q yields the result. ∎

If $H(K,L)$ is finitely generated, its Euler characteristic, called the *Euler characteristic of* (K,L), is denoted by $\chi(K,L)$.

15 COROLLARY *If $K - L$ is finite and if α_q equals the number of q-simplexes of $K - L$, then*

$$\chi(K,L) = \Sigma(-1)^q \alpha_q$$

PROOF If $K - L$ is finite, $C_q(K)/C_q(L)$ is a free group of rank α_q. The result follows from theorem 14. ∎

4 SINGULAR HOMOLOGY

In this section we define a natural transformation from the ordered chain complex to the singular chain complex of its space. This will be shown in Sec. 4.6 to be a chain equivalence for every simplicial complex K. We also give a proof, based on acyclic models, that homotopic continuous maps induce chain-homotopic chain maps on the singular chain complexes. There is then a computation of $H_0(X)$ in terms of the path components of X. The final result is that the subcomplex of the singular chain complex generated by "small" singular simplexes is chain equivalent to the whole singular chain complex.[1]

Let K be a simplicial complex. Given an ordered q-simplex $(v_0,v_1, \ldots ,v_q)$ of K, there is a singular q-simplex in $|K|$ which is the linear map $\Delta^q \to |K|$ sending p_i to v_i for $0 \le i \le q$. This imbeds $\Delta(K)$ in $\Delta(|K|)$, and we define an augmentation-preserving chain map

$$\nu\colon \Delta(K) \to \Delta(|K|)$$

to send $(v_0,v_1, \ldots ,v_q)$ to the linear singular simplex defined above. Then ν is a natural chain map from the covariant functor $\Delta(\cdot)$ to the covariant functor $\Delta(|\cdot|)$ on the category of simplicial complexes. It will be shown in Sec. 4.6 that ν is a natural chain equivalence. We prove now that it is a chain equivalence for the complex $\bar{s}$ of an arbitrary simplex s.

1 LEMMA *Let X be a star-shaped subset of some euclidean space. Then the reduced singular complex of X is chain contractible.*

[1] Our treatment is similar to that in S. Eilenberg, Singular homology theory, *Annals of Mathematics*, vol. 45, pp. 407–447 (1944).

PROOF Without loss of generality, X may be assumed to be star-shaped from the origin. We define a homomorphism $\tau: \mathbf{Z} \to \Delta_0(X)$ with $\tau(1)$ equal to the singular simplex $\Delta^0 \to X$ which is the constant map to 0. Then $\varepsilon \circ \tau = 1_{\mathbf{Z}}$. We define a chain homotopy $D: \Delta(X) \to \Delta(X)$ from $1_{\Delta(X)}$ to $\tau \circ \varepsilon$. If $\sigma: \Delta^q \to X$ is a singular q-simplex in X, let $D(\sigma): \Delta^{q+1} \to X$ be the singular $(q+1)$-simplex in X defined by the equation

$$D(\sigma)(tp_0 + (1-t)\alpha) = (1-t)\sigma(\alpha)$$

for $\alpha \in |p_1, \ldots, p_{q+1}|$ and $t \in I$. If $q > 0$, then $(D(\sigma))^{(0)} = \sigma$, and for $0 \le i \le q$, $(D(\sigma))^{(i+1)} = D(\sigma^{(i)})$. If $q = 0$, then $(D(\sigma))^{(0)} = \sigma$ and $(D(\sigma))^{(1)} = \tau(1)$. Therefore

$$\partial D + D\partial = 1_{\Delta(X)} - \tau \circ \varepsilon$$

and $D: 1_{\Delta(X)} \simeq \tau \circ \varepsilon$. By lemma 4.3.2, $\tilde{\Delta}(X)$ is chain contractible. ∎

2 COROLLARY *For any simplex s the chain map ν induces an isomorphism of the ordered homology group of $\bar{s}$ with the singular homology group of $|\bar{s}|$.*

PROOF Because ν preserves augmentation, ν induces a homomorphism $\tilde{\nu}_*$ from $\tilde{H}(\Delta(\bar{s}))$ to $\tilde{H}(|\bar{s}|)$, and under the isomorphism of lemma 4.3.1, $\nu_* = \tilde{\nu}_* \oplus 1_{\mathbf{Z}}$. By corollary 4.3.7, $\tilde{H}(\Delta(\bar{s})) = 0$. By lemma 1 and corollary 4.2.3, $\tilde{H}(|\bar{s}|) = 0$. Therefore ν_* is an isomorphism. ∎

We use lemma 1 to prove that if $f_0, f_1: X \to Y$ are homotopic, then $\Delta(f_0), \Delta(f_1): \Delta(X) \to \Delta(Y)$ are chain homotopic. We prove this first for the maps $h_0, h_1: X \to X \times I$, where $h_0(x) = (x,0)$ and $h_1(x) = (x,1)$.

3 THEOREM *The maps $h_0, h_1: X \to X \times I$ induce naturally chain-homotopic chain maps*

$$\Delta(h_0) \simeq \Delta(h_1): \Delta(X) \to \Delta(X \times I)$$

PROOF Let $\Delta'(X) = \Delta(X \times I)$. Then Δ and Δ' are covariant functors from the category of topological spaces to the category of augmented chain complexes and $\Delta(h_0)$ and $\Delta(h_1)$ are natural chain maps preserving augmentation from Δ to Δ'. Since Δ is free with models $\{\Delta^q\}$ and

$$\tilde{\Delta}'(\Delta^q) = \tilde{\Delta}(\Delta^q \times I)$$

is acyclic, by lemma 1, it follows from theorem 4.3.3 that $\Delta(h_0)$ and $\Delta(h_1)$ are naturally chain homotopic. ∎

This special case implies the general result.

4 COROLLARY *If $f_0, f_1: X \to Y$ are homotopic, then*

$$\Delta(f_0) \simeq \Delta(f_1): \Delta(X) \to \Delta(Y)$$

PROOF Let $F: X \times I \to Y$ be a homotopy from f_0 to f_1. Then $f_0 = Fh_0$ and $f_1 = Fh_1$. Therefore, using theorem 3,

$$\Delta(f_0) = \Delta(F)\Delta(h_0) \simeq \Delta(F)\Delta(h_1) = \Delta(f_1) \quad ∎$$

Since Δ^q is path connected for every q, any singular simplex $\sigma\colon \Delta^q \to X$ maps Δ^q to some path component of X. Hence, if $\{X_j\}$ is the set of path components of X, then $\Delta(X) = \bigoplus \Delta(X_j)$. By theorem 4.1.6, we have the following theorem.

5 THEOREM *The singular homology group of a space is the direct sum of the singular homology groups of its path components.* ∎

Because Δ^q is compact, every singular simplex $\sigma\colon \Delta^q \to X$ maps Δ^q into some compact subset of X. Hence, if $\{X_\alpha\}$ is the collection of compact subsets of X directed by inclusion, then $\Delta(X) = \lim_{\to} \Delta(X_\alpha)$. By theorem 4.1.7, we have our next result.

6 THEOREM *The singular homology group of a space is isomorphic to the direct limit of the singular homology groups of its compact subsets.* ∎

We now compute the 0-dimensional homology group of a space.

7 LEMMA *If X is a nonempty path-connected topological space, then $\tilde{H}_0(X) = 0$.*

PROOF Let x_0 be a fixed point of X. For any point $x \in X$ there is a path ω_x from x_0 to x. Because Δ^1 is homeomorphic to I, ω_x corresponds to a singular 1-simplex $\sigma_x\colon \Delta^1 \to X$ such that $\sigma_x{}^{(0)} = x$ and $\sigma_x{}^{(1)} = x_0$. A singular 0-simplex in X is identified with a point of X. Therefore a 0-chain (that is, a 0-cycle) of X is a sum $\Sigma n_x x$, where $n_x = 0$ except for a finite set of x's. Since $\varepsilon(\Sigma n_x x) = \Sigma n_x$, we see that if $\varepsilon(\Sigma n_x x) = 0$ [that is, if $\Sigma n_x x \in \tilde{\Delta}_0(X)$], then

$$\partial(\Sigma n_x \sigma_x) = \Sigma n_x x - (\Sigma n_x) x_0 = \Sigma n_x x$$

Therefore $\tilde{H}_0(X) = 0$. ∎

8 COROLLARY *For any topological space X, $H_0(X)$ is a free group whose rank equals the number of nonempty components of X.*

PROOF If X is empty, $H_0(X) = 0$, and the result is valid in this case. If X is nonempty and path connected, it follows from lemmas 7 and 4.3.1 that $H_0(X) \approx \mathbf{Z}$. The general result now follows from theorem 5. ∎

If A is a subspace of X, there is a *relative singular homology group* $H(X,A) = \{H_q(X,A) = H_q(\Delta(X)/\Delta(A))\}$ *of X modulo A*. $H(X,\varnothing) = H(X)$ is called the *absolute singular homology group of X*. The relative homology group is a covariant functor from the category of topological pairs to the category of graded groups. We show that this functor can be regarded as defined on the homotopy category of pairs.

9 THEOREM *If $f_0, f_1\colon (X,A) \to (Y,B)$ are homotopic, then*

$$f_{0*} = f_{1*}\colon H(X,A) \to H(Y,B)$$

PROOF Let $F\colon (X \times I, A \times I) \to (Y,B)$ be a homotopy from f_0 to f_1. Then $f_0 = F\bar{h}_0$ and $f_1 = F\bar{h}_1$, where $\bar{h}_0, \bar{h}_1\colon (X,A) \to (X \times I, A \times I)$ are defined by

$\bar{h}_0(x) = (x,0)$ and $\bar{h}_1(x) = (x,1)$. By theorem 3, there is a natural chain homotopy D: $\Delta(h_0) \simeq \Delta(h_1)$, where h_0, h_1: $X \to X \times I$ are maps defined by $\bar{h}_0$ and $\bar{h}_1$. Because D is natural, $D(\Delta(A)) \subset \Delta(A \times I)$. For $i = 0$ or 1 there is a commutative diagram

$$\begin{array}{ccccc} \Delta(A) & \subset & \Delta(X) & \to & \Delta(X)/\Delta(A) \\ {\scriptstyle \Delta(h_i)}\downarrow & & {\scriptstyle \Delta(h_i)}\downarrow & & \downarrow{\scriptstyle \Delta(\bar{h}_i)} \\ \Delta(A \times I) & \subset & \Delta(X \times I) & \to & \Delta(X \times I)/\Delta(A \times I) \end{array}$$

and a chain homotopy $\bar{D}$: $\Delta(\bar{h}_0) \simeq \Delta(\bar{h}_1)$ is obtained by passing to the quotient with D. By theorem 4.2.2,

$$\bar{h}_{0*} = \bar{h}_{1*}: H(X,A) \to H(X \times I, A \times I)$$

Then

$$f_{0*} = F_* \bar{h}_{0*} = F_* \bar{h}_{1*} = f_{1*} \quad \blacksquare$$

If $H_q(X,A)$ is finitely generated, its rank is called the *qth Betti number of* (X,A) and the orders of its finite cyclic summands given by the structure theorem are called the *qth torsion coefficients of* (X,A). If $H(X,A)$ is finitely generated, its Euler characteristic is called the *Euler characteristic of* (X,A), denoted by $\chi(X,A)$.

The remainder of this section is directed toward a proof that the subcomplex of the singular chain complex generated by small singular simplexes is chain equivalent to the singular chain complex. We begin by defining a subdivision chain map in singular theory. A singular simplex σ: $\Delta^q \to \Delta^n$ is said to be *linear* if $\sigma(\sum t_i p_i) = \sum t_i \sigma(p_i)$ for $t_i \in I$ with $\sum t_i = 1$. If σ is linear, so is $\sigma^{(i)}$ for $0 \leq i \leq q$. Therefore the set of linear simplexes in Δ^n generates a subcomplex $\Delta'(\Delta^n) \subset \Delta(\Delta^n)$.

A linear simplex σ in Δ^n is completely determined by the points $\sigma(p_i)$. If $x_0, x_1, \ldots, x_q \in \Delta^n$, we write $(x_0,x_1, \ldots ,x_q)$ to denote the linear simplex σ: $\Delta^q \to \Delta^n$ such that $\sigma(p_i) = x_i$. With this notation, it is clear that

$$\partial(x_0, \ldots, x_q) = \sum(-1)^i(x_0, \ldots ,\hat{x}_i, \ldots ,x_q)$$

Furthermore, the identity map ξ_n: $\Delta^n \subset \Delta^n$ is the linear simplex $\xi_n = (p_0,p_1, \ldots ,p_n)$.

Let b_n be the barycenter of Δ^n (that is, $b_n = \sum(1/(n+1))p_i$. For $q \geq 0$ a homomorphism

$$\beta_n: \Delta'_q(\Delta^n) \to \Delta'_{q+1}(\Delta^n)$$

is defined by the formula

$$\beta_n(x_0, \ldots ,x_q) = (b_n,x_0, \ldots ,x_q)$$

Let τ: $\mathbf{Z} \to \Delta'_0(\Delta^n)$ be defined by $\tau(1) = (b_n)$. Direct computation shows that

10

$$\beta_n: 1_{\Delta'(\Delta^n)} \simeq \tau \circ \varepsilon \quad \blacksquare$$

For every topological space X we define an augmentation-preserving chain map

$$sd\colon \Delta(X) \to \Delta(X)$$

and a chain homotopy

$$D\colon \Delta(X) \to \Delta(X)$$

from sd to $1_{\Delta(X)}$, both of which are functorial in X. That is, if $f\colon X \to Y$, there are commutative squares

$$\begin{array}{ccc} \Delta(X) & \xrightarrow{sd} & \Delta(X) \\ {\scriptstyle \Delta(f)}\downarrow & & \downarrow{\scriptstyle \Delta(f)} \\ \Delta(Y) & \xrightarrow{sd} & \Delta(Y) \end{array} \qquad \begin{array}{ccc} \Delta(X) & \xrightarrow{D} & \Delta(X) \\ {\scriptstyle \Delta(f)}\downarrow & & \downarrow{\scriptstyle \Delta(f)} \\ \Delta(Y) & \xrightarrow{D} & \Delta(Y) \end{array}$$

Both sd and D are defined on q-chains by induction on q. If c is a 0-chain, we define $sd(c) = c$ and $D(c) = 0$. Assume sd and D defined on q-chains for $0 \le q < n$, where $n \ge 1$. We define sd and D on the universal singular n-simplex $\xi_n\colon \Delta^n \subset \Delta^n$ by the formulas

$$\begin{aligned} sd(\xi_n) &= \beta_n(sd\,\partial(\xi_n)) \\ D(\xi_n) &= \beta_n(sd\,(\xi_n) - \xi_n - D\partial(\xi_n)) \end{aligned}$$

For any singular n-simplex $\sigma\colon \Delta^n \to X$ we define

$$\begin{aligned} sd(\sigma) &= \Delta(\sigma)(sd(\xi_n)) \\ D(\sigma) &= \Delta(\sigma)(D(\xi_n)) \end{aligned}$$

Then sd and D have all the requisite properties.

If X is a metric space and $c = \Sigma n_\sigma \sigma$ is a singular q-chain of X, we define

$$\text{mesh } c = \sup \{\text{diam } \sigma(\Delta^q) \mid n_\sigma \ne 0\}$$

11 LEMMA *Let Δ^n have a linear metric and let c be a linear q-chain of Δ^n. Then*

$$\text{mesh } (sd\,c) \le \frac{q}{q+1} \text{ mesh } c$$

PROOF The proof is based on induction on q, using the inductive definition of sd. It suffices to show that if $\sigma = (x_0,x_1, \ldots ,x_q)$ is a linear q-simplex of Δ^n, then mesh $(sd\,\sigma) \le (q/(q+1))$ mesh σ. If $b = \Sigma\,(1/(q+1))x_i$, a computation similar to that of lemma 3.3.12 shows that the distance from b to any convex combination of the points $x_0, x_1, \ldots, x_q$ is less than or at most equal to $(q/(q+1))$ mesh $(x_0, \ldots ,x_q)$. Therefore

$$\text{mesh } (sd\,\sigma) \le \sup \left(\frac{q}{q+1} \text{ mesh } \sigma, \text{ mesh } (sd\,\partial\sigma)\right)$$

By induction

$$\text{mesh } (sd\ \partial\sigma) \le \frac{q-1}{q} \text{ mesh } \partial\sigma$$
$$\le \frac{q}{q+1} \text{ mesh } \sigma$$

which yields the result. ∎

We next define augmentation-preserving chain maps

$$sd^m\colon \Delta(X) \to \Delta(X)$$

for $m \ge 0$ by induction

$$sd^0 = 1_{\Delta(X)} \qquad \text{and} \qquad sd^m = sd(sd^{m-1}) \qquad m \ge 1$$

Then, from lemma 11, we obtain the following result.

12 COROLLARY *Let Δ^n have a linear metric and let $c \in \Delta'_q(\Delta^n)$. Then*

$$\text{mesh } (sd^m\ c) \le [q/(q+1)]^m \text{ mesh } c \quad ∎$$

Let $\mathfrak{A} = \{A\}$ be a collection of subsets of a topological space X and let $\Delta(\mathfrak{A})$ be the subcomplex of $\Delta(X)$ generated by singular q-simplexes $\sigma\colon \Delta^q \to X$ such that $\sigma(\Delta^q) \subset A$ for some $A \in \mathfrak{A}$ [if $\sigma(\Delta^q) \subset A$, then $\sigma^{(i)}(\Delta^{q-1}) \subset A$, and so $\Delta(\mathfrak{A})$ is a subcomplex of $\Delta(X)$]. Because sd and D are natural, $sd(\Delta(\mathfrak{A})) \subset \Delta(\mathfrak{A})$ and $D(\Delta(\mathfrak{A})) \subset \Delta(\mathfrak{A})$.

13 LEMMA *Let $\mathfrak{A} = \{A\}$ be such that $X = \bigcup \{\text{int } A \mid A \in \mathfrak{A}\}$. For any singular q-simplex σ of X there is $m \ge 0$ such that $sd^m\ \sigma \in \Delta(\mathfrak{A})$.*

PROOF Because $X = \bigcup\{\text{int } A \mid A \in \mathfrak{A}\}$, $\Delta^q = \bigcup\{\sigma^{-1}(\text{int } A) \mid A \in \mathfrak{A}\}$. Let Δ^q be metrized by a linear metric and let $\lambda > 0$ be a Lebesgue number for the open covering $\{\sigma^{-1}(\text{int } A) \mid A \in \mathfrak{A}\}$ of Δ^q relative to this metric. Choose $m \ge 0$ so that $[q/(q+1)]^m \text{ diam } \Delta^q \le \lambda$. By corollary 12, mesh $(\text{sd}^m\ \xi_q) \le \lambda$. Therefore every singular simplex of $sd^m\ \xi_q$ maps into $\sigma^{-1}(\text{int } A)$ for some $A \in \mathfrak{A}$. Then $sd^m\ \sigma = \Delta(\sigma)\ sd^m\ \xi_q$ is a chain in $\Delta(\mathfrak{A})$. ∎

We are now ready to prove the chain equivalence mentioned earlier.

14 THEOREM *Let $\mathfrak{A} = \{A\}$ be such that $X = \bigcup\{\text{int } A \mid A \in \mathfrak{A}\}$. Then the inclusion map $\Delta(\mathfrak{A}) \subset \Delta(X)$ is a chain equivalence.*

PROOF For each singular simplex σ in X let $m(\sigma)$ be the smallest nonnegative integer such that $sd^{m(\sigma)}\sigma \in \Delta(\mathfrak{A})$. Such an integer $m(\sigma)$ exists by lemma 13, and it is clear that $m(\sigma) = 0$ if and only if $\sigma \in \Delta(\mathfrak{A})$. Furthermore, $m(\sigma^{(i)}) \le m(\sigma)$ for $0 \le i \le \deg \sigma$.

Define $\bar{D}\colon \Delta(X) \to \Delta(X)$ by $\bar{D}(\sigma) = \sum_{0 \le j \le m(\sigma)-1} D\ sd^j(\sigma)$. Then $\bar{D}(\sigma) = 0$ if and only if $\sigma \in \Delta(\mathfrak{A})$. Also

$$\begin{aligned}
\partial\bar{D}(\sigma) &= \Sigma sd^{j+1}(\sigma) - \Sigma sd^j(\sigma) - \Sigma D\ sd^j(\partial\sigma) \\
&= sd^{m(\sigma)}(\sigma) - \sigma - \textstyle\sum_{0 \le j \le m(\sigma)-1} \sum_i (-1)^i D\ sd^j(\sigma^{(i)}) \\
\bar{D}\partial(\sigma) &= \textstyle\sum_i (-1)^i \sum_{0 \le j \le m(\sigma^{(i)})-1} D\ sd^j(\sigma^{(i)})
\end{aligned}$$

Therefore

$$\sigma + \partial\bar{D}(\sigma) + \bar{D}\partial(\sigma) = \textstyle\sum_i (-1)^i \sum_{m(\sigma^{(i)})\leq j\leq m(\sigma)-1} D\, sd^j(\sigma^{(i)}) + sd^{m(\sigma)}(\sigma)$$

is in $\Delta(\mathcal{U})$. Define τ: $\Delta(X) \to \Delta(\mathcal{U})$ by $\tau(\sigma) = \sigma + \partial\bar{D}(\sigma) + \bar{D}\partial(\sigma)$. Then τ is a chain map preserving augmentation. Clearly, if i: $\Delta(\mathcal{U}) \subset \Delta(X)$, then $\tau \circ i = 1_{\Delta(\mathcal{U})}$ and $\bar{D}$: $i \circ \tau \simeq 1_{\Delta(X)}$. Therefore $[\tau] = [i]^{-1}$, and i is a chain equivalence. ∎

5 EXACTNESS

In this section we consider the relations among the homology groups of C', C, and C/C', where C' is a subcomplex of C. A concise way of summarizing these relations is by means of the concept of exact sequence. The basic result is the existence of an exact sequence connecting the homology of C', C, and C/C'.

A three-term sequence of abelian groups and homomorphisms

$$G' \xrightarrow{\alpha} G \xrightarrow{\beta} G''$$

is said to be *exact at* G if ker β = im α. A sequence of abelian groups and homomorphisms indexed by integers (which may or may not terminate at either or both ends)

$$\cdots \to G_{n+1} \xrightarrow{\alpha_{n+1}} G_n \xrightarrow{\alpha_n} G_{n-1} \to \cdots$$

is said to be an *exact sequence* if every three-term subsequence of consecutive groups is exact at its middle group. Note that an exact sequence terminating at one end with a trivial group can be extended indefinitely on that end to an exact sequence by adjoining trivial groups and homomorphisms.

A *short exact sequence of abelian groups,* written

$$0 \to G' \xrightarrow{\alpha} G \xrightarrow{\beta} G'' \to 0$$

is a five-term exact sequence whose end groups are trivial. In such a short exact sequence α is a monomorphism and β is an epimorphism whose kernel is $\alpha(G')$. Therefore α is an isomorphism of G' with the subgroup $\alpha(G') \subset G$, and β induces an isomorphism from the quotient group $G/\alpha(G')$ to G''. The group G is called an *extension of G' by G''.*

Given an exact sequence

$$\cdots \to G_{n+1} \xrightarrow{\alpha_{n+1}} G_n \xrightarrow{\alpha_n} G_{n-1} \to \cdots$$

let G'_n = ker α_n = im α_{n+1}. Then the given sequence gives rise to short exact sequences

$$0 \to G'_n \to G_n \to G'_{n-1} \to 0$$

for every G_n not on one or the other end of the original sequence, and the composite $G_n \to G'_{n-1} \to G_{n-1}$ equals α_n.

A *homomorphism* γ from one sequence $\{G_n \xrightarrow{\alpha_n} G_{n-1}\}$ to another $\{H_n \xrightarrow{\beta_n} H_{n-1}\}$ with the same set of indices (that is, of the same length) is a sequence $\{\gamma_n\colon G_n \to H_n\}$ of homomorphisms such that the following diagram is commutative:

$$\begin{array}{ccccccccc}
\cdots & \to & G_{n+1} & \xrightarrow{\alpha_{n+1}} & G_n & \xrightarrow{\alpha_n} & G_{n-1} & \to & \cdots \\
 & & {\scriptstyle\gamma_{n+1}}\downarrow & & {\scriptstyle\gamma_n}\downarrow & & \downarrow{\scriptstyle\gamma_{n-1}} & & \\
\cdots & \to & H_{n+1} & \xrightarrow{\beta_{n+1}} & H_n & \xrightarrow{\beta_n} & H_{n-1} & \to & \cdots
\end{array}$$

There is a category of exact sequences with the same set of indices. In particular, there is a category of short exact sequences, and also a category of exact sequences (indexed by all the integers).

Note that a sequence of abelian groups and homomorphisms

$$\cdots \to C_{n+1} \xrightarrow{\partial_{n+1}} C_n \xrightarrow{\partial_n} C_{n-1} \to \cdots$$

is a chain complex if and only if im $\partial_{n+1} \subset$ ker ∂_n for all n. This is half of the condition of exactness at C_n. For a chain complex C, the group $H_n(C) =$ ker ∂_n/im ∂_{n+1} is a measure of the nonexactness of the sequence at C_n. Thus a chain complex is an exact sequence if and only if its graded homology group is trivial. In any case, the fact that the homology group measures the nonexactness of the chain complex suggests that there should be some relation between homology and exactness, and this is indeed so.

A *short exact sequence of chain complexes*, written

$$0 \to C' \xrightarrow{\alpha} C \xrightarrow{\beta} C'' \to 0$$

is a five-term sequence of chain complexes and chain maps such that for all q there is a short exact sequence of abelian groups

$$0 \to C'_q \xrightarrow{\alpha_q} C_q \xrightarrow{\beta_q} C''_q \to 0$$

A homomorphism from one short exact sequence of chain complexes to another consists of a commutative diagram of chain maps

$$\begin{array}{ccccccccc}
0 & \to & C' & \xrightarrow{\alpha} & C & \xrightarrow{\beta} & C'' & \to & 0 \\
 & & {\scriptstyle\tau'}\downarrow & & {\scriptstyle\tau}\downarrow & & \downarrow{\scriptstyle\tau''} & & \\
0 & \to & \bar{C}' & \xrightarrow{\bar{\alpha}} & \bar{C} & \xrightarrow{\bar{\beta}} & \bar{C}'' & \to & 0
\end{array}$$

There is a category of short exact sequences of chain complexes and homomorphisms.

1 EXAMPLE Let C' be a subcomplex of a chain complex C and let $i\colon C' \subset C$ and $j\colon C \to C/C'$ be the inclusion and projection chain maps, respectively. There is a short exact sequence of chain complexes

$$0 \to C' \xrightarrow{i} C \xrightarrow{j} C/C' \to 0$$

Given a subcomplex $\bar{C}' \subset \bar{C}$ and a chain map $\tau: C \to \bar{C}$ such that $\tau(C') \subset \bar{C}'$, there is a homomorphism

$$\begin{array}{ccccccccc} 0 & \to & C' & \xrightarrow{i} & C & \xrightarrow{j} & C/C' & \to & 0 \\ & & {\scriptstyle\tau'}\downarrow & & {\scriptstyle\tau}\downarrow & & \downarrow{\scriptstyle\tau''} & & \\ 0 & \to & \bar{C}' & \xrightarrow{\bar{i}} & \bar{C} & \xrightarrow{\bar{j}} & \bar{C}/\bar{C}' & \to & 0 \end{array}$$

where $\tau' = \tau \mid C'$ and τ'' is induced from τ by passing to the quotient.

2 EXAMPLE If C is an augmented chain complex, there is a short exact sequence of chain complexes

$$0 \to \tilde{C} \to C \xrightarrow{\varepsilon} \mathbf{Z} \to 0$$

There is a covariant functor C from the category of simplicial pairs to the category of short exact sequences of chain complexes which assigns to (K,L) the short exact sequence

$$0 \to C(L) \to C(K) \to C(K)/C(L) \to 0$$

Similarly, there is a covariant functor Δ from the category of topological pairs to the category of short exact sequences of chain complexes which assigns to (X,A) the short exact sequence

$$0 \to \Delta(A) \to \Delta(X) \to \Delta(X)/\Delta(A) \to 0$$

There is also a covariant functor Δ from the category of simplicial pairs to the category of short exact sequences of chain complexes which assigns to (K,L) the short exact sequence

$$0 \to \Delta(L) \to \Delta(K) \to \Delta(K)/\Delta(L) \to 0$$

Then μ is a natural transformation from Δ to C and ν is a natural transformation from Δ to $\Delta(|\cdot|)$ (both natural transformations in the category of short exact sequences of chain complexes).

We define covariant functors H', H, and H'' from the category of short exact sequences of chain complexes

$$0 \to C' \xrightarrow{\alpha} C \xrightarrow{\beta} C'' \to 0$$

to the category of graded groups such that H', H, and H'' map the above sequence into $H(C')$, $H(C)$, and $H(C'')$, respectively.

3 LEMMA *On the category of short exact sequences of chain complexes*

$$0 \to C' \xrightarrow{\alpha} C \xrightarrow{\beta} C'' \to 0$$

there is a natural transformation $\partial_*: H'' \to H'$ such that if $\{z''\} \in H(C'')$, then $\partial_* \{z''\} = \{\alpha^{-1}\partial\beta^{-1}z''\} \in H(C')$.

PROOF There is a commutative diagram

$$\begin{array}{ccccccccc}
0 & \to & C'_{q+1} & \xrightarrow{\alpha} & C_{q+1} & \xrightarrow{\beta} & C''_{q+1} & \to & 0 \\
& & \partial' \downarrow & & \partial \downarrow & & \downarrow \partial'' & & \\
0 & \to & C'_q & \xrightarrow{\alpha} & C_q & \xrightarrow{\beta} & C''_q & \to & 0 \\
& & \partial' \downarrow & & \partial \downarrow & & \downarrow \partial'' & & \\
0 & \to & C'_{q-1} & \xrightarrow{\alpha} & C_{q-1} & \xrightarrow{\beta} & C''_{q-1} & \to & 0
\end{array}$$

in which each row is a short exact sequence of groups. If z'' is a q-cycle of C'', let $c \in C_q$ be such that $\beta(c) = z''$. Then

$$\beta(\partial c) = \partial''\beta(c) = \partial''z'' = 0$$

Therefore there is a unique $c' \in C'_{q-1}$ such that $\alpha(c') = \partial c$. Then

$$\alpha(\partial'c') = \partial\alpha(c') = \partial\partial c = 0$$

Because α is a monomorphism, $\partial'c' = 0$. Hence c' is a $(q - 1)$-cycle of C'.

We show that the homology class of c' in C' depends only on the homology class of z'' in C'', which will prove that there is a well-defined homomorphism $\partial_* \{z''\} = \{c'\}$. Let $c_1 \in C_q$ be such that $\beta(c_1) \sim z''$. Then there is $d'' \in C''_{q+1}$ such that $\beta(c_1) = \beta(c) + \partial''d''$. Choose $d \in C_{q+1}$ such that $\beta(d) = d''$. Then

$$\beta(c_1) = \beta(c) + \partial''\beta(d) = \beta(c + \partial d)$$

Therefore there is a $d' \in C'_q$ such that $c_1 = c + \partial d + \alpha(d')$, and

$$\partial c_1 = \partial c + \partial\alpha(d') = \alpha(c') + \alpha(\partial'd') = \alpha(c' + \partial'd')$$

Hence $\alpha^{-1}(\partial c_1) = c' + \partial'd' \sim c'$ and $\{\alpha^{-1}(\partial c_1)\} = \{\alpha^{-1}(\partial c)\}$, showing that ∂_* is well-defined.

To prove that ∂_* is a natural transformation, assume given a commutative diagram of chain maps

$$\begin{array}{ccccccccc}
0 & \to & C' & \xrightarrow{\alpha} & C & \xrightarrow{\beta} & C'' & \to & 0 \\
& & \tau' \downarrow & & \tau \downarrow & & \downarrow \tau'' & & \\
0 & \to & \bar{C}' & \xrightarrow{\bar{\alpha}} & \bar{C} & \xrightarrow{\bar{\beta}} & \bar{C}'' & \to & 0
\end{array}$$

where the horizontal rows are short exact sequences. Then

$$\begin{aligned}
\tau'_* \partial_* \{z''\} &= \tau'_* \{\alpha^{-1}\partial\beta^{-1}z''\} = \{\tau'\alpha^{-1}\partial\beta^{-1}z''\} \\
&= \{\bar{\alpha}^{-1}\tau\partial\beta^{-1}z''\} = \{\bar{\alpha}^{-1}\bar{\partial}\bar{\beta}^{-1}\tau''z''\} = \bar{\partial}_* \tau''_* \{z''\} \quad \blacksquare
\end{aligned}$$

The natural transformation ∂_* is called the *connecting homomorphism* for homology because of its importance in the following *exactness theorem.*

4 THEOREM *There is a covariant functor from the category of short exact sequences of chain complexes to the category of exact sequences of groups which assigns to a short exact sequence*

$$0 \to C' \xrightarrow{\alpha} C \xrightarrow{\beta} C'' \to 0$$

the sequence

$$\cdots \xrightarrow{\partial_*} H_q(C') \xrightarrow{\alpha_*} H_q(C) \xrightarrow{\beta_*} H_q(C'') \xrightarrow{\partial_*} H_{q-1}(C') \xrightarrow{\alpha_*} \cdots$$

PROOF The sequence of homology groups is functorial on short exact sequences because ∂_* is a natural transformation. It only remains to verify that it is an exact sequence. This entails a proof of exactness at $H_q(C')$, $H_q(C)$, and $H_q(C'')$, each exactness requiring two inclusion relations. Therefore the proof of exactness has six parts. We shall prove exactness at $H_q(C'')$ and leave the other parts of the proof to the reader.

(*a*) im $\beta_* \subset$ ker ∂_*. Let $\{z\} \in H_q(C)$. Then

$$\partial_* \beta_* \{z\} = \partial_* \{\beta(z)\} = \{\alpha^{-1}\partial\beta^{-1}\beta(z)\} = \{\alpha^{-1}\partial z\} = \{\alpha^{-1}(0)\} = 0$$

(*b*) ker $\partial_* \subset$ im β_*. Let $\{z''\} \in$ ker ∂_*. Then there is $c \in C_q$ such that $\beta(c) = z''$ and $\alpha^{-1}\partial(c) = \partial'(d')$ for some $d' \in C'_q$. The difference $c - \alpha(d') \in C_q$ is such that

$$\partial(c - \alpha(d')) = \partial c - \alpha(\partial' d') = 0$$

Hence $\{c - \alpha(d')\} \in H_q(C)$ and

$$\beta_* \{c - \alpha(d')\} = \{\beta(c) - \beta\alpha(d')\} = \{z''\} \quad \blacksquare$$

Combining theorem 4 with example 2, we again obtain lemma 4.3.1. As an example of the utility of exactness, note that the following corollary is immediate from theorem 4.

5 COROLLARY *Given a short exact sequence of chain complexes*

$$0 \to C' \xrightarrow{\alpha} C \xrightarrow{\beta} C'' \to 0$$

(*a*) *C' is acyclic if and only if $\beta_* : H(C) \approx H(C'')$.*
(*b*) *C is acyclic if and only if $\partial_* : H(C'') \approx H(C')$.*
(*c*) *C'' is acyclic if and only if $\alpha_* : H(C') \approx H(C)$.* $\blacksquare$

In (*b*) above it should be noted that ∂_* has degree -1. It follows from corollary 5 that if two of the chain complexes C', C, and C'' are acyclic, so is the third.

6 COROLLARY *Given an exact sequence of abelian groups*

$$\cdots \to G_{n+1} \xrightarrow{\alpha_{n+1}} G_n \xrightarrow{\alpha_n} G_{n-1} \to \cdots$$

and a subsequence

$$\cdots \to G'_{n+1} \xrightarrow{\alpha'_{n+1}} G'_n \xrightarrow{\alpha'_n} G'_{n-1} \to \cdots$$

(that is, $G'_n \subset G_n$ and $\alpha'_n = \alpha_n \mid G'_n$), the subsequence is exact if and only if the quotient sequence

$$\cdots \to G_n/G'_n \to G_{n-1}/G'_{n-1} \to \cdots$$

is exact.

PROOF Let C be the chain complex consisting of the original exact sequence and let C' be the subcomplex consisting of the subsequence. Then the quotient chain complex C/C' is the quotient sequence. Because C is an exact sequence, C is acyclic, and $\partial_*\colon H_q(C/C') \approx H_{q-1}(C')$. Therefore C' is exact [that is, $H(C') = 0$] if and only if C/C' is exact [that is, $H(C/C') = 0$]. ∎

7 THEOREM *The direct limit of exact sequences is exact.*

PROOF Each exact sequence is an acyclic chain complex. The direct limit is also a chain complex, and it is acyclic, by theorem 4.1.7. Therefore the limit sequence is exact. ∎

This result is false if direct limit is replaced by inverse limit, because the homology functor fails to commute with inverse limits.

Let K be a simplicial complex and let $L_1 \subset L_2 \subset K$. By the Noether isomorphism theorem, there is a short exact sequence of chain complexes

$$0 \to C(L_2)/C(L_1) \xrightarrow{i} C(K)/C(L_1) \xrightarrow{j} C(K)/C(L_2) \to 0$$

By theorem 4, there is an exact sequence

$$\cdots \xrightarrow{\partial_*} H_q(L_2,L_1) \xrightarrow{i_*} H_q(K,L_1) \xrightarrow{j_*} H_q(K,L_2) \xrightarrow{\partial_*} H_{q-1}(L_2,L_1) \xrightarrow{i_*} \cdots$$

where i_* is induced by $i\colon (L_2,L_1) \subset (K,L_1)$, j_* is induced by $j\colon (K,L_1) \subset (K,L_2)$, and ∂_* is the connecting homomorphism. This sequence is called the *homology sequence of the triple* (K,L_2,L_1). It is functorial on triples. If $L_1 = \varnothing$, the resulting exact sequence

$$\cdots \xrightarrow{\partial_*} H_q(L_2) \xrightarrow{i_*} H_q(K) \xrightarrow{j_*} H_q(K,L_2) \xrightarrow{\partial_*} H_{q-1}(L_2) \xrightarrow{i_*} \cdots$$

is called the *homology sequence of the pair* (K,L_2). It is functorial on pairs.

Because there is an inclusion map of the triple $(K,L_2,\varnothing)$ into the triple (K,L_2,L_1), the next result follows.

8 LEMMA *The connecting homomorphism* $\partial_*\colon H_q(K,L_2) \to H_{q-1}(L_2,L_1)$ *of the triple* (K,L_2,L_1) *is the composite*

$$H_q(K,L_2) \xrightarrow{\partial_*} H_{q-1}(L_2) \xrightarrow{k_*} H_{q-1}(L_2,L_1)$$

of the connecting homomorphism of the pair (K,L_2) *followed by the homomorphism induced by* $k\colon (L_2,\varnothing) \subset (L_2,L_1)$. ∎

If L is a nonempty subcomplex of a simplicial complex, $\tilde{C}(L) \subset \tilde{C}(K)$, and by the Noether isomorphism theorem, $\tilde{C}(K)/\tilde{C}(L) \approx C(K)/C(L)$. Therefore there is a short exact sequence of chain complexes

$$0 \to \tilde{C}(L) \xrightarrow{i} \tilde{C}(K) \xrightarrow{j} C(K)/C(L) \to 0$$

The corresponding exact sequence

$$\cdots \xrightarrow{\partial_*} \tilde{H}_q(L) \xrightarrow{i_*} \tilde{H}_q(K) \xrightarrow{j_*} H_q(K,L) \xrightarrow{\partial_*} \tilde{H}_{q-1}(L) \xrightarrow{i_*} \cdots$$

is called the *reduced homology sequence of the pair* (K,L). It is not defined if $L = \varnothing$, because $C(L)$ has no augmentation in this case.

In the same way, there is a *singular homology sequence of a triple* (X,A,B) and *of a pair* (X,A). If A is nonempty, there is also a *reduced homology sequence of* (X,A). All these sequences are exact, and the analogue of lemma 8 is valid relating the connecting homomorphism of a triple to the connecting homomorphism of a pair.

9 LEMMA *Let s be an n-simplex. Then*

$$H_q(\bar{s},\dot{s}) \approx \begin{cases} 0 & q \neq n \\ \mathbf{Z} & q = n \end{cases}$$

PROOF $C_q(\dot{s}) = C_q(\bar{s})$ if $q \neq n$. Therefore $[C(\bar{s})/C(\dot{s})]_q = 0$ if $q \neq n$, and $[C(\bar{s})/C(\dot{s})]_n \approx \mathbf{Z}$. ∎

Because $\tilde{H}(\bar{s}) = 0$, by corollary 4.3.7, it follows from the exactness of the reduced homology sequence of $(\bar{s},\dot{s})$ that $\partial_*\colon H_q(\bar{s},\dot{s}) \approx \tilde{H}_{q-1}(\dot{s})$ for all q. Therefore we have the next result.

10 COROLLARY *If s is an n-simplex, then*

$$\tilde{H}_q(\dot{s}) \approx \begin{cases} 0 & q \neq n-1 \\ \mathbf{Z} & q = n-1 \end{cases} \quad ∎$$

We conclude by proving the following *five lemma* (so named because of the five-term exact sequences involved in its formulation).

11 LEMMA *Given a commutative diagram of abelian groups and homomorphisms*

$$\begin{array}{ccccccccc} G_5 & \xrightarrow{\alpha_5} & G_4 & \xrightarrow{\alpha_4} & G_3 & \xrightarrow{\alpha_3} & G_2 & \xrightarrow{\alpha_2} & G_1 \\ \downarrow{\scriptstyle\gamma_5} & & \downarrow{\scriptstyle\gamma_4} & & \downarrow{\scriptstyle\gamma_3} & & \downarrow{\scriptstyle\gamma_2} & & \downarrow{\scriptstyle\gamma_1} \\ H_5 & \xrightarrow{\beta_5} & H_4 & \xrightarrow{\beta_4} & H_3 & \xrightarrow{\beta_3} & H_2 & \xrightarrow{\beta_2} & H_1 \end{array}$$

in which each row is exact and γ_1, γ_2, γ_4, and γ_5 are isomorphisms, then γ_3 is an isomorphism.

PROOF The proof is straightforward. To show that γ_3 is a monomorphism, assume $\gamma_3(g_3) = 0$. Then $\gamma_2\alpha_3(g_3) = \beta_3\gamma_3(g_3) = 0$. Therefore $\alpha_3(g_3) = 0$. Hence there is $g_4 \in G_4$ such that $\alpha_4(g_4) = g_3$. Then $\beta_4\gamma_4(g_4) = 0$, and there is $h_5 \in H_5$ such that $\beta_5(h_5) = \gamma_4(g_4)$. There is $g_5 \in G_5$ with $\gamma_5(g_5) = h_5$. Then $\gamma_4(\alpha_5(g_5)) = \gamma_4(g_4)$, and so $g_4 = \alpha_5(g_5)$. Then $g_3 = \alpha_4\alpha_5(g_5) = 0$.

To show that γ_3 is an epimorphism let $h_3 \in H_3$. There is $g_2 \in G_2$ such that $\gamma_2(g_2) = \beta_3(h_3)$. Then $\gamma_1\alpha_2(g_2) = \beta_2\beta_3(h_3) = 0$. Therefore $\alpha_2(g_2) = 0$, and there is $g_3 \in G_3$ such that $\alpha_3(g_3) = g_2$. Then $\beta_3(h_3 - \gamma_3(g_3)) = 0$, and there is $h_4 \in H_4$ such that $\beta_4(h_4) = h_3 - \gamma_3(g_3)$. Let $g_4 \in G_4$ be such that $\gamma_4(g_4) = h_4$. Then $g_3 + \alpha_4(g_4) \in G_3$ and $\gamma_3(g_3 + \alpha_4(g_4)) = \gamma_3(g_3) + \beta_4(h_4) = h_3$. ∎

Note that to prove γ_3 a monomorphism we merely needed γ_2 and γ_4 to be monomorphisms and γ_5 to be an epimorphism, and to prove γ_3 an epimorphism we merely needed γ_2 and γ_4 to be epimorphisms and γ_1 to be a monomorphism. This type of proof is called *diagram chasing* and will be omitted in the future.

We shall have several occasions to use the five lemma. We mention the following as a typical example. For any simplicial pair (K,L) the natural transformation μ from the ordered homology theory induces a homomorphism of the corresponding exact sequences

$$\begin{array}{ccccccccc} \cdots & \to & H_q(\Delta(L)) & \to & H_q(\Delta(K)) & \to & H_q(\Delta(K)/\Delta(L)) & \to & \cdots \\ & & \downarrow{\scriptstyle \mu_*} & & \downarrow{\scriptstyle \mu_*} & & \downarrow{\scriptstyle \mu_*} & & \\ \cdots & \to & H_q(L) & \to & H_q(K) & \to & H_q(K,L) & \to & \cdots \end{array}$$

By theorem 4.3.8, μ_* is an isomorphism on the absolute groups. It follows from the five lemma that it is also an isomorphism on the relative groups.

12 COROLLARY *For any simplicial pair (K,L) the natural transformation μ induces an isomorphism from the ordered homology sequence of (K,L) to the oriented homology sequence of (K,L).* ∎

6 MAYER-VIETORIS SEQUENCES

There is an exact sequence which relates the homology of the union of two sets to the homology of each of the sets and to the homology of their intersection. This sequence provides an inductive procedure for computing the homology of spaces which are built from pieces whose homology is known. We shall define this exact sequence as well as its analogue involving relative homology groups, and use them to prove that the natural transformation ν from $\Delta(K)$ to $\Delta(|K|)$ is a chain equivalence for any simplicial complex K.

Let K_1 and K_2 be subcomplexes of a simplicial complex K. Then $K_1 \cap K_2$ and $K_1 \cup K_2$ are subcomplexes of K, and $C(K_1), C(K_2) \subset C(K)$. Clearly $C(K_1 \cap K_2) = C(K_1) \cap C(K_2)$ and $C(K_1) + C(K_2) = C(K_1 \cup K_2)$. Let $i_1\colon K_1 \cap K_2 \subset K_1$, $i_2\colon K_1 \cap K_2 \subset K_2$, $j_1\colon K_1 \subset K_1 \cup K_2$, and $j_2\colon K_2 \subset K_1 \cup K_2$. Then we have a short exact sequence of chain complexes

$$0 \to C(K_1 \cap K_2) \xrightarrow{i} C(K_1) \oplus C(K_2) \xrightarrow{j} C(K_1 \cup K_2) \to 0$$

where $i(c) = (C(i_1)c, -C(i_2)c)$ and $j(c_1,c_2) = C(j_1)c_1 + C(j_2)c_2$. The corresponding exact sequence of homology groups

$$\cdots \xrightarrow{\partial_*} H_q(K_1 \cap K_2) \xrightarrow{i_*} H_q(K_1) \oplus H_q(K_2) \xrightarrow{j_*} H_q(K_1 \cup K_2) \xrightarrow{\partial_*} H_{q-1}(K_1 \cap K_2) \xrightarrow{i_*} \cdots$$

is called the *Mayer-Vietoris sequence* of the subcomplexes K_1 and K_2. The homomorphisms i_* and j_* in the Mayer-Vietoris sequence are described by means of homomorphisms induced by inclusion maps by

$$i_* z = (i_{1*} z, -i_{2*} z) \qquad \text{and} \qquad j_* (z_1,z_2) = j_{1*} z_1 + j_{2*} z_2$$

for $z \in H(K_1 \cap K_2)$, $z_1 \in H(K_1)$, and $z_2 \in H(K_2)$.

If $K_1 \cap K_2 \neq \varnothing$, there is a commutative diagram of abelian groups and homomorphisms

$$\begin{array}{ccccccccc} 0 & \to & C_0(K_1 \cap K_2) & \xrightarrow{i} & C_0(K_1) \oplus C_0(K_2) & \xrightarrow{j} & C_0(K_1 \cup K_2) & \to & 0 \\ & & \varepsilon\downarrow & & \varepsilon \oplus \varepsilon\downarrow & & \downarrow\varepsilon & & \\ 0 & \to & \mathbf{Z} & \xrightarrow{\alpha} & \mathbf{Z} \oplus \mathbf{Z} & \xrightarrow{\beta} & \mathbf{Z} & \to & 0 \end{array}$$

where $\alpha(n) = (n,-n)$ and $\beta(n,m) = n + m$. Since the rows are exact and the vertical homomorphisms are epimorphisms, it follows from corollary 4.5.6 that there is an exact sequence of the kernels

$$0 \to \tilde{C}_0(K_1 \cap K_2) \xrightarrow{i} \tilde{C}_0(K_1) \oplus \tilde{C}_0(K_2) \xrightarrow{j} \tilde{C}_0(K_1 \cup K_2) \to 0$$

and so there is a short exact sequence of chain complexes

$$0 \to \tilde{C}(K_1 \cap K_2) \xrightarrow{i} \tilde{C}(K_1) \oplus \tilde{C}(K_2) \xrightarrow{j} \tilde{C}(K_1 \cup K_2) \to 0$$

The corresponding exact sequence of reduced homology groups

$$\cdots \xrightarrow{\partial_*} \tilde{H}_q(K_1 \cap K_2) \xrightarrow{i_*} \tilde{H}_q(K_1) \oplus \tilde{H}_q(K_2) \xrightarrow{j_*} \tilde{H}_q(K_1 \cup K_2) \xrightarrow{\partial_*} \cdots$$

is called the *reduced Mayer-Vietoris sequence of* K_1 *and* K_2.

If (K_1,L_1) and (K_2,L_2) are simplicial pairs in K, there is also a short exact sequence

$$0 \to C(L_1 \cap L_2) \to C(L_1) \oplus C(L_2) \to C(L_1 \cup L_2) \to 0$$

which is a subsequence of the short exact sequence

$$0 \to C(K_1 \cap K_2) \to C(K_1) \oplus C(K_2) \to C(K_1 \cup K_2) \to 0$$

It follows from corollary 4.5.6 that the quotient sequence is a short exact sequence of chain complexes

$$0 \to C(K_1 \cap K_2)/C(L_1 \cap L_2) \to C(K_1)/C(L_1) \oplus C(K_2)/C(L_2) \to C(K_1 \cup K_2)/C(L_1 \cup L_2) \to 0$$

The corresponding exact sequence of homology groups

$$\cdots \xrightarrow{\partial_*} H_q(K_1 \cap K_2, L_1 \cap L_2) \xrightarrow{i_*} H_q(K_1,L_1) \oplus H_q(K_2,L_2) \xrightarrow{j_*} H_q(K_1 \cup K_2, L_1 \cup L_2) \xrightarrow{\partial_*} \cdots$$

is called the *relative Mayer-Vietoris sequence of* (K_1,L_1) *and* (K_2,L_2).

The relative Mayer-Vietoris sequence specializes to the exact sequence of a triple or a pair. In fact, given a triple (K,L_1,L_2), the relative Mayer-Vietoris sequence of (K,L_2) and (L_1,L_1) is easily seen to be the homology sequence of the triple (K,L_1,L_2) as defined in Sec. 4.5. In case $L_2 = \varnothing$, the relative Mayer-Vietoris sequence of $(K,\varnothing)$ and (L_1,L_1) is the homology sequence of the pair (K,L_1).

An inclusion map $(K_1,L_1) \subset (K_2,L_2)$ is called an *excision map* if $K_1 - L_1 = K_2 - L_2$. The exactness of the Mayer-Vietoris sequence is closely related (in fact, equivalent) to the following *excision property.*

1 THEOREM *Any excision map between simplicial pairs induces an isomorphism on homology.*

PROOF If $(K_1,L_1) \subset (K_2,L_2)$ is an excision map, then $K_2 = K_1 \cup L_2$ and $L_1 = K_1 \cap L_2$. By the Noether isomorphism theorem,

$$C(K_1)/C(L_1) \approx [C(K_1) + C(L_2)]/C(L_2) = C(K_2)/C(L_2) \quad \blacksquare$$

For the ordered chain complex it is still true that if K_1 and K_2 are subcomplexes of some simplicial complex, then $\Delta(K_1 \cup K_2) = \Delta(K_1) + \Delta(K_2)$. Therefore all the above results remain valid if the oriented homology is replaced throughout by the ordered homology.

An inclusion map $(X_1,A_1) \subset (X_2,A_2)$ between topological pairs is called an *excision map* if $X_1 - A_1 = X_2 - A_2$. It is *not* true that every excision map induces an isomorphism of the singular homology groups. Neither is it true that there is an exact Mayer-Vietoris sequence of any two subsets X_1 and X_2 of a topological space.

2 EXAMPLE Let $f\colon \mathbf{R} \to \mathbf{R}$ be defined by

$$f(x) = \begin{cases} \sin \dfrac{1}{x} & x > 0 \\ 0 & x \leq 0 \end{cases}$$

and let $X_1 = \{(x,y) \in \mathbf{R}^2 \mid y \geq f(x) \text{ or } x = 0, |y| \leq 1\}$ and $X_2 = \{(x,y) \in \mathbf{R}^2 \mid y \leq f(x) \text{ or } x = 0, |y| \leq 1\}$. Then X_1 and X_2 are closed path-connected subsets of $\mathbf{R}^2$ such that $X_1 \cup X_2 = \mathbf{R}^2$ and $X_1 \cap X_2$ consists of two path components. Therefore there is no homomorphism $\tilde{H}_1(X_1 \cup X_2) \to \tilde{H}_0(X_1 \cap X_2)$ which will make the sequence

$$\tilde{H}_1(X_1 \cup X_2) \to \tilde{H}_0(X_1 \cap X_2) \to \tilde{H}_0(X_1) \oplus \tilde{H}_0(X_2)$$

exact at $\tilde{H}_0(X_1 \cap X_2)$ [the ends are both trivial, but $\tilde{H}_0(X_1 \cap X_2) \neq 0$].

We can, however, develop a Mayer-Vietoris sequence in singular homology for certain subsets X_1 and X_2 of a topological space. Let X_1 and X_2 be subsets of some space. $\{X_1,X_2\}$ is said to be an *excisive couple of subsets* if the inclusion chain map $\Delta(X_1) + \Delta(X_2) \subset \Delta(X_1 \cup X_2)$ induces an isomorphism of homology. Our next result follows from theorem 4.4.14.

3 THEOREM *If* $X_1 \cup X_2 = \operatorname{int}_{X_1 \cup X_2} X_1 \cup \operatorname{int}_{X_1 \cup X_2} X_2$, *then* $\{X_1,X_2\}$ *is an excisive couple.* $\blacksquare$

In particular, if $A \subset X$, then $\{X,A\}$ is always an excisive couple. The relation between an excisive couple $\{X_1,X_2\}$ and excision maps is expressed as follows.

4 THEOREM $\{X_1,X_2\}$ *is an excisive couple if and only if the excision map* $(X_1,X_1 \cap X_2) \subset (X_1 \cup X_2,X_2)$ *induces an isomorphism of singular homology.*

PROOF We have a commutative diagram of chain maps induced by inclusions

$$\begin{tikzcd}
\Delta(X_1)/\Delta(X_1 \cap X_2) \arrow[rr, "\Delta(j)"] \arrow[dr, "i"'] & & \Delta(X_1 \cup X_2)/\Delta(X_2) \\
& {[\Delta(X_1) + \Delta(X_2)]/\Delta(X_2)} \arrow[ur, "i'"'] &
\end{tikzcd}$$

where j is the excision map $j\colon (X_1, X_1 \cap X_2) \subset (X_1 \cup X_2, X_2)$. By the Noether isomorphism theorem, i is an isomorphism; therefore $j_* = i'_* i_*$ is an isomorphism if and only if i'_* is an isomorphism. Using the exactness of the homology sequence of a pair and the five lemma, i'_* is an isomorphism if and only if the inclusion map $\Delta(X_1) + \Delta(X_2) \subset \Delta(X_1 \cup X_2)$ induces an isomorphism of homology, which is by definition equivalent to the condition that $\{X_1,X_2\}$ be an excisive couple. ■

This yields the following *excision property for singular theory.*

5 COROLLARY *Let $U \subset A \subset X$ be such that $\bar{U} \subset \operatorname{int} A$. Then the excision map $(X - U, A - U) \subset (X,A)$ induces an isomorphism of singular homology.*

PROOF The hypothesis $\bar{U} \subset \operatorname{int} A$ implies $\operatorname{int}(X - U) \supset X - \bar{U} \supset X - \operatorname{int} A$. By theorem 3, $\{A, X - U\}$ is an excisive couple, and the result follows from this and from theorem 4. ■

For any subsets X_1 and X_2 of a space, $\Delta(X_1 \cap X_2) = \Delta(X_1) \cap \Delta(X_2)$, and there is a short exact sequence of singular chain complexes

$$0 \to \Delta(X_1 \cap X_2) \xrightarrow{i} \Delta(X_1) \oplus \Delta(X_2) \xrightarrow{j} \Delta(X_1) + \Delta(X_2) \to 0$$

This yields an exact sequence

$$\cdots \xrightarrow{\partial_*} H_q(X_1 \cap X_2) \xrightarrow{i_*} H_q(X_1) \oplus H_q(X_2) \xrightarrow{j_*} H_q(\Delta(X_1) + \Delta(X_2)) \xrightarrow{\partial_*} H_{q-1}(X_1 \cap X_2) \to \cdots$$

If $\{X_1,X_2\}$ is an excisive couple, the group $H_q(\Delta(X_1) + \Delta(X_2))$ can be replaced by the group $H_q(X_1 \cup X_2)$, and the resulting exact sequence is

$$\cdots \xrightarrow{\partial_*} H_q(X_1 \cap X_2) \xrightarrow{i_*} H_q(X_1) \oplus H_q(X_2) \xrightarrow{j_*} H_q(X_1 \cup X_2) \xrightarrow{\partial_*} H_{q-1}(X_1 \cap X_2) \to \cdots$$

where $i_*(z) = (i_{1*}z, -i_{2*}z)$ and $j_*(z_1,z_2) = j_{1*}z_1 + j_{2*}z_2$ for $z \in H(X_1 \cap X_2)$, $z_1 \in H(X_1)$, and $z_2 \in H(X_2)$. This is the *Mayer-Vietoris sequence of singular theory of an excisive couple* $\{X_1,X_2\}$. Similarly, if $X_1 \cap X_2 \neq \varnothing$, there is a reduced Mayer-Vietoris sequence of $\{X_1,X_2\}$.

If (X_1,A_1) and (X_2,A_2) are pairs in a space X, we say that $\{(X_1,A_1), (X_2,A_2)\}$ is an *excisive couple of pairs* if $\{X_1,X_2\}$ and $\{A_1,A_2\}$ are both excisive couples of subsets. In this case it follows from the five lemma that the map induced by inclusion

$$[\Delta(X_1) + \Delta(X_2)]/[\Delta(A_1) + \Delta(A_2)] \to [\Delta(X_1 \cup X_2)]/[\Delta(A_1 \cup A_2)]$$

induces an isomorphism of homology. Hence, if $\{(X_1,A_1), (X_2,A_2)\}$ is an excisive couple of pairs, there is an exact sequence

$$\cdots \xrightarrow{\partial_*} H_q(X_1 \cap X_2, A_1 \cap A_2) \xrightarrow{i_*} H_q(X_1,A_1) \oplus H_q(X_2,A_2) \xrightarrow{j_*} H_q(X_1 \cup X_2, A_1 \cup A_2) \xrightarrow{\partial_*} \cdots$$

called the *relative Mayer-Vietoris sequence of* $\{(X_1,A_1), (X_2,A_2)\}$.

The relative Mayer-Vietoris sequence specializes to the exact sequence of a triple (or a pair). In fact, given a triple (X,A,B), $\{(X,B), (A,A)\}$ is always an excisive couple of pairs, and the relative Mayer-Vietoris sequence of $\{(X,B), (A,A)\}$ is the homology sequence of the triple (X,A,B).

We use the Mayer-Vietoris sequence to compute the singular homology of a sphere.

6 THEOREM *For* $n \geq 0$

$$\tilde{H}_q(S^n) \approx \begin{cases} 0 & q \neq n \\ \mathbf{Z} & q = n \end{cases}$$

PROOF Let p and p' be distinct points of S^n. Because $S^n - p$ and $S^n - p'$ are contractible (each being homeomorphic to $\mathbf{R}^n$), $\tilde{H}(S^n - p) = 0 = \tilde{H}(S^n - p')$. Since $S^n - p$ and $S^n - p'$ are open subsets of S^n, it follows from theorem 3 that $\{S^n - p, S^n - p'\}$ is an excisive couple. From the exactness of the corresponding Mayer-Vietoris sequence, it follows that

$$\partial_*: \tilde{H}_q(S^n) \approx \tilde{H}_{q-1}(S^n - (p \cup p'))$$

Because $S^n - (p \cup p')$ has the same homotopy type as S^{n-1}, there is an isomorphism $\tilde{H}_{q-1}(S^n - (p \cup p')) \approx \tilde{H}_{q-1}(S^{n-1})$, and the result follows by induction and the trivial verification that for $n = 0$ the theorem is valid. ■

We now show that a couple consisting of polyhedral subsets of a polyhedron is excisive.

7 LEMMA *Let K_1 and K_2 be subcomplexes of a simplicial complex K. Then $\{|K_1|,|K_2|\}$ is an excisive couple.*

PROOF Let V be a neighborhood of $|K_1 \cap K_2|$ in $|K_1|$ having $|K_1 \cap K_2|$ as a strong deformation retract (such a V exists, by corollary 3.3.11). There is a commutative diagram

$$\begin{tikzcd}
\cdots \arrow[r] & H_q(|K_1 \cap K_2|) \arrow[r] \arrow[d, "i_*"'] & H_q(|K_1|) \arrow[r] \arrow[d, "1"'] & H_q(|K_1|, |K_1 \cap K_2|) \arrow[r] \arrow[d, "j_*"'] & \cdots \\
\cdots \arrow[r] & H_q(V) \arrow[r] & H_q(|K_1|) \arrow[r] & H_q(|K_1|,V) \arrow[r] & \cdots
\end{tikzcd}$$

Because $i: |K_1 \cap K_2| \subset V$ is a homotopy equivalence, $i_*: H(|K_1 \cap K_2|) \approx H(V)$. By the five lemma, $j_*: H(|K_1|, |K_1 \cap K_2|) \approx H(|K_1|,V)$.

Also, $V \cup |K_2|$ is a neighborhood of $|K_2|$ in $|K_1 \cup K_2|$ having $|K_2|$ as a strong deformation retract. Therefore a similar proof shows that

$$j'_*: H(|K_1 \cup K_2|, |K_2|) \approx H(|K_1 \cup K_2|, V \cup |K_2|)$$

By theorem 4, $\{|K_1|, |K_2|\}$ is an excisive couple if and only if the excision map $(|K_1|, |K_1 \cap K_2|) \subset (|K_1 \cup K_2|, |K_2|)$ induces an isomorphism of homology. In view of the isomorphisms j_* and j'_*, this will be so if and only if the excision map $(|K_1|,V) \subset (|K_1 \cup K_2|, V \cup |K_2|)$ induces an isomorphism of homology. Again by theorem 4, this is equivalent to the condition that $\{|K_1|, V \cup |K_2|\}$ be an excisive couple. This is so by theorem 3, since $|K_2| \subset \text{int}\,(V \cup |K_2|)$ and $|K_1| - |K_2| \subset \text{int}\,|K_1|$. ∎

8 THEOREM *For any simplicial pair (K,L) the natural transformation ν induces an isomorphism of the ordered homology sequence of (K,L) onto the singular homology sequence of $(|K|,|L|)$.*

PROOF It suffices to prove that for any simplicial complex K, $\nu_*: H(\Delta(K)) \approx H(|K|)$, because the theorem will follow from this and the five lemma. We prove this first for finite simplicial complexes by induction on the number of simplexes. If K contains one simplex, then $K = \bar{s}$, where s is a 0-simplex, and the result follows from corollary 4.4.2.

Assume the result inductively for simplicial complexes with fewer than m simplexes, where $m > 1$, and let K contain exactly m simplexes. Let s be a simplex of K of maximum dimension and let L be the subcomplex of K consisting of all simplexes other than s. Then $K = L \cup \bar{s}$ and $\dot{s} = L \cap \bar{s}$. Because L has exactly $m - 1$ simplexes, ν_* is an isomorphism $H(\Delta(L)) \approx H(|L|)$ and an isomorphism $H(\Delta(\dot{s})) \approx H(|\dot{s}|)$. By corollary 4.4.2, $\nu_*: H(\Delta(\bar{s})) \approx H(|\bar{s}|)$. By the exactness of the ordered Mayer-Vietoris sequence of L and $\bar{s}$ and the Mayer-Vietoris sequence of singular theory for $|L|$ and $|\bar{s}|$ (which exists, by lemma 7), it follows from the five lemma that $\nu_*: H(\Delta(K)) \approx H(|K|)$.

For infinite simplicial complexes K let $\{K_\alpha\}$ be the family of finite subcomplexes of K directed by inclusion. It follows from theorem 4.3.11 that $H(\Delta(K)) \approx \lim_\rightarrow H(\Delta(K_\alpha))$ and from theorem 4.4.6 that $H(|K|) \approx \lim_\rightarrow H(|K_\alpha|)$. The theorem now holds for K because ν_* is natural. ∎

We show next that for free chain complexes a chain map is a chain equivalence if and only if it induces an isomorphism in homology. First we establish an exact sequence containing the homomorphism induced by a chain map.

9 LEMMA *Let $\tau: C \to C'$ be a chain map and let $\bar{C}$ be the mapping cone of τ. There is an exact sequence*

$$\cdots \to H_{q+1}(\bar{C}) \to H_q(C) \xrightarrow{\tau_*} H_q(C') \to H_q(\bar{C}) \to \cdots$$

PROOF Let $\alpha: C' \to \bar{C}$ be the chain map defined by $\alpha(c) = (0,c)$. Then α imbeds C' as a subcomplex of $\bar{C}$ and the quotient complex $\bar{C}/C'$ is such that $(\bar{C}/C')_q \approx C_{q-1}$; the boundary operator of $\bar{C}/C'$ corresponds to the negative of the boundary operator of C under this isomorphism. The desired exact sequence is then obtained from the exact homology sequence of the short exact sequence of chain complexes

$$0 \to C' \xrightarrow{\alpha} \bar{C} \to \bar{C}/C' \to 0$$

by replacing $H_q(\bar{C}/C')$ by $H_{q-1}(C)$ and verifying that the connecting homomorphism $\partial_*: H_{q+1}(\bar{C}/C') \to H_q(C')$ corresponds to $\tau_*: H_q(C) \to H_q(C')$. ∎

10 THEOREM *If C and C′ are free chain complexes, a chain map τ: $C \to C'$ is a chain equivalence if and only if τ_*: $H(C) \approx H(C')$.*

PROOF By corollary 4.2.11, τ is a chain equivalence if and only if $\bar{C}$ is acyclic. By lemma 9 and corollary 4.5.5, $\bar{C}$ is acyclic if and only if $\tau_*: H(C) \approx H(C')$. ∎

Because $\Delta(K)/\Delta(L)$ and $\Delta(|K|)/\Delta(|L|)$ are free chain complexes, we have the following result.

11 COROLLARY *For any simplicial pair (K,L), ν is a chain equivalence of $\Delta(K)/\Delta(L)$ with $\Delta(|K|)/\Delta(|L|)$.* ∎

If φ: $K_1 \to K_2$ is a simplicial map, there is a commutative diagram

$$\begin{tikzcd} H(K_1) \arrow[d, "\varphi_*"'] & H(\Delta(K_1)) \arrow[l, "\mu_*"', "\approx"] \arrow[r, "\nu_*", "\approx"'] \arrow[d, "\Delta(\varphi)_*"] & H(|K_1|) \arrow[d, "|\varphi|_*"] \\ H(K_2) & H(\Delta(K_2)) \arrow[l, "\mu_*"', "\approx"] \arrow[r, "\nu_*", "\approx"'] & H(|K_2|) \end{tikzcd}$$

In particular, if K' is a subdivision of K and φ: $K' \to K$ is a simplicial approximation to the identity $|K'| \subset |K|$, then

$$|\varphi| \simeq 1_{|K|} \qquad \text{and} \qquad |\varphi|_* = 1_{H(|K|)}$$

From the commutativity of the above diagram we obtain our next result.

12 THEOREM *Let K′ be a subdivision of K and let φ: $K' \to K$ be a simplicial approximation to the identity map $|K'| \subset |K|$. Then*

$$\varphi_*: H(K') \approx H(K) \quad ∎$$

By theorem 10, $C(\varphi)$: $C(K') \to C(K)$ is a chain equivalence. It will be useful to construct a chain map $C(K) \to C(K')$ which is a chain homotopy inverse of $C(\varphi)$. If K' is a subdivision of K, an augmentation-preserving chain map τ: $C(K) \to C(K')$ is called a *subdivision chain map* if τ: $C(L) \subset C(K' \mid L)$ for every subcomplex $L \subset K$ [that is, if τ is a natural chain map from C to $C(K' \mid \cdot\,)$ on $\mathcal{C}(K)$].

13 THEOREM *If K′ is a subdivision of K, there exist subdivision chain maps τ: $C(K) \to C(K')$. If φ: $K' \to K$ is a simplicial approximation to the identity $|K'| \subset |K|$, then $\tau_* = \varphi_*^{-1}$: $H(K) \approx H(K')$.*

PROOF If s is any simplex of K, then $\tilde{C}(K' \mid \bar{s})$ is acyclic [because $\tilde{H}(K' \mid \bar{s}) \approx \tilde{H}(|\bar{s}|) = 0$]. Hence, on the category $\mathcal{C}(K)$ of subcomplexes of K with models $\mathfrak{M}(K) = \{\bar{s} \mid s \in K\}$, the functor C is free and $C(K' \mid \cdot\,)$ is acyclic. It follows from theorem 4.3.3 that there exist natural chain maps τ from C to $C(K' \mid \cdot\,)$ preserving augmentation.

If τ is any subdivision chain map and φ: $K' \to K$ is a simplicial approximation to the identity map $|K'| \subset K$, the composite

$$C(\varphi)\tau\colon C(K) \to C(K)$$

is a natural chain map over $\mathcal{C}(K)$ from C to C preserving augmentation. Since C is free and acyclic with models $\mathfrak{M}(K)$, it follows from theorem 4.3.3 that $C(\varphi)\tau \simeq 1_{C(K)}$. Therefore $\varphi_* \tau_* = 1_{H(K)}$. Since, by theorem 12, φ_* is an isomorphism, $\tau_* = \varphi_*^{-1}$. ∎

7 SOME APPLICATIONS OF HOMOLOGY

In this section we use homology for some of the applications mentioned earlier. We shall show that euclidean spaces of different dimensions are not homeomorphic, and also that S^n is not a retract of E^{n+1} (which is easily seen to be equivalent to the Brouwer fixed-point theorem). This leads to the general consideration of fixed points of maps, and we prove the Lefschetz fixed-point theorem. Finally, we shall consider separation properties of the sphere. Proofs are given of Brouwer's generalization of the Jordan curve theorem and of the invariance of domain.

1 THEOREM *If $n \neq m$, S^n and S^m are not of the same homotopy type.*

PROOF By theorem 4.6.6, $\tilde H_n(S^n) \neq 0$ and $\tilde H_n(S^m) = 0$. ∎

2 COROLLARY *If $n \neq m$, $\mathbf{R}^n$ and $\mathbf{R}^m$ are not homeomorphic.*

PROOF If $\mathbf{R}^n$ and $\mathbf{R}^m$ were homeomorphic, their one-point compactifications S^n and S^m would also be homeomorphic, in contradiction to theorem 1. ∎

In corollary 2 both $\mathbf{R}^n$ and $\mathbf{R}^m$ are contractible. Therefore they have the same homotopy type and cannot be distinguished by their homology groups. To distinguish them it was necessary to consider associated spaces having nonisomorphic homology. We chose to consider their one-point compactifications, but another proof could have been based on the fact that $\mathbf{R}^n$ minus a point has the same homotopy type as S^{n-1}.

These two results are applications of homology to the problem of classifying spaces up to topological equivalence. Our next application is to an extension problem.

3 LEMMA *Let (X,A) be a pair such that A is a retract of X. Then*

$$H(X) \approx H(A) \oplus H(X,A)$$

PROOF Given $i\colon A \subset X$ and $j\colon (X,\varnothing) \subset (X,A)$ and a retraction $r\colon X \to A$, then $ri = 1_A$. Therefore $r_* i_* = 1_{H(A)}$ and i_* is a monomorphism of $H(A)$ onto a direct summand of $H(X)$. The other summand is the kernel of r_*. From the exactness of the homology sequence of (X,A)

$$\cdots \to H_q(X,A) \xrightarrow{\partial_*} H_{q-1}(A) \xrightarrow{i_*} H_{q-1}(X) \xrightarrow{j_*} H_{q-1}(X,A) \xrightarrow{\partial_*} \cdots$$

because ker $i_* = 0$, ∂_* is the trivial map. Therefore j_* is an epimorphism. Since ker $j_* = \text{im } i_*$, j_* induces an isomorphism of ker r_* onto $H(X,A)$. ■

Note that lemma 3 is still valid if A is a weak retract of X.

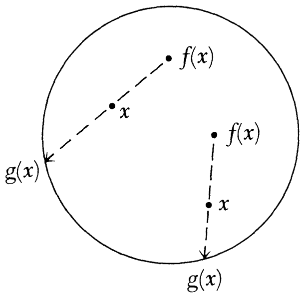
4 COROLLARY *For $n \geq 0$, S^n is not a retract of E^{n+1}.*

PROOF By theorem 4.6.6, $\tilde{H}_n(S^n) \neq 0$, but because E^{n+1} is contractible, $\tilde{H}_n(E^{n+1}) = 0$. Therefore $H(S^n)$ is not isomorphic to a direct summand of $H(E^{n+1})$. ■

This implies the following *Brouwer fixed-point theorem.*

5 THEOREM *For $n \geq 0$ every continuous map from E^n to itself has a fixed point.*

PROOF For $n = 0$ there is nothing to prove. For $n > 0$ let $f\colon E^n \to E^n$ be continuous. If f has no fixed point, define a map $g\colon E^n \to S^{n-1}$ by $g(x)$ equal to the unique point of S^{n-1} on the ray from $f(x)$ to x, as shown in the figure.



Then g is a retraction from E^n to S^{n-1}, in contradiction to corollary 4. ■

We have, in fact, proved that corollary 4 implies theorem 5. The converse is also true, for if $r\colon E^{n+1} \to S^n$ were a retraction, the map $f\colon E^{n+1} \to E^{n+1}$ defined by $f(x) = -r(x)$ would have no fixed points.

There is an interesting generalization of theorem 5 which contains a criterion for showing that a certain map from X to itself has a fixed point even if not every map of X to itself has fixed points. This generalization also illustrates another type of application of homology in that it is based on an algebraic count of the number of fixed points, the algebraic count being formulated in homological terms. This type of application of homology occurs frequently. Generally it involves a set of singularities of X of a certain type (for example, the set of fixed points of a map $X \to X$, the set of discontinuties of a function $X \to Y$, the set of self-intersections of a local homeomorphism $X \to \mathbf{R}^n$, etc.) and measures the singular set by means of a homology class associated to it.

Let C be a finitely generated graded group and let $h\colon C \to C$ be an endomorphism of C of degree 0. The *Lefschetz number* $\lambda(h)$ is defined by the formula

$$\lambda(h) = \Sigma(-1)^q Tr(h_q)$$

where $h_q\colon C_q \to C_q$ is the endomorphism defined by h in degree q. The following *Hopf trace formula* equates the Lefschetz numbers of a chain map and its induced homology homomorphism.

6 THEOREM *Let C be a finitely generated chain complex and let $\tau\colon C \to C$ be a chain map. Then*

$$\lambda(\tau) = \lambda(\tau_*)$$

PROOF The proof is similar to the proof of the corresponding statement about the Euler characteristic (theorem 4.3.14), the Euler characteristic being the Lefschetz number of the identity map, with theorem 4.13 of the Introduction used in place of theorem 4.12. Details are left to the reader. ∎

Let $f\colon X \to X$ be a map, where X has finitely generated homology. The *Lefschetz number of f*, denoted by $\lambda(f)$, is defined to be the Lefschetz number of the homomorphism $f_*\colon H(X) \to H(X)$ induced by f. It counts the algebraic number of fixed homology classes of f_*. The following *Lefschetz fixed-point theorem* shows that $\lambda(f) \neq 0$ is a sufficient condition for f to have a fixed point.

7 THEOREM *Let X be a compact polyhedron and let $f\colon X \to X$ be a map. If $\lambda(f) \neq 0$, then f has a fixed point.*

PROOF We assume that f has no fixed point and prove $\lambda(f) = 0$. Without loss of generality, we may assume $X = |L|$ for some finite simplicial complex L. Because $|L|$ is a compact metric space, if f has no fixed point, there is $a > 0$ such that $d(\alpha,f(\alpha)) \geq a$ for all $\alpha \in |L|$. Let K be a subdivision of L with mesh $K < a/3$ and let K' be a subdivision of K for which there exists a simplicial map $\varphi\colon K' \to K$ which is a simplicial approximation to $f\colon |K| \to |K|$. Since $|\varphi|(\alpha)$ and $f(\alpha)$ belong to some simplex of K, $d(|\varphi|(\alpha),f(\alpha)) < a/3$ for $\alpha \in |K|$. If s is any simplex of K, $|s|$ is disjoint from $|\varphi|(|s|)$, for if $\alpha \in |s|$ is equal to $|\varphi|(\beta)$ for $\beta \in |s|$, then

$$d(\beta,f(\beta)) \leq d(\beta,\alpha) + d(|\varphi|(\beta),f(\beta)) < 2a/3$$

in contradiction to the choice of a.

Let $\tau\colon C(K) \to C(K')$ be a subdivision chain map (which exists, by theorem 4.6.13). Then $C(\varphi)\tau\colon C(K) \to C(K)$ is a chain map. If σ is an oriented q-simplex on a q-simplex s of K, then $C(\varphi)\tau(\sigma)$ is a q-chain on the largest subcomplex of K disjoint from s. Therefore $C(\varphi)\tau(\sigma)$ is a q-chain having coefficient 0 on σ. Since this is so for every σ, all the coefficients summed in forming $Tr(C(\varphi)\tau)_q$ are zero and $Tr((C(\varphi)\tau)_q) = 0$ for all q, which implies $\lambda(C(\varphi)\tau) = 0$. By theorem 6, $\lambda((C(\varphi)\tau)_*) = 0$. Let $\varphi'\colon K' \to K$ be a simplicial approximation to the identity map $|K'| \subset |K|$. There is a commutative diagram

$$\begin{tikzcd}
H(K) & H(K') \arrow[l, "\varphi'_*"'] \arrow[r, "\varphi_*"] & H(K) \\
H(\Delta(K)) \arrow[u, "\approx"'] \arrow[d, "\approx"] & H(\Delta(K')) \arrow[l, "\Delta(\varphi')_*"'] \arrow[r, "\Delta(\varphi)_*"] \arrow[u, "\approx"'] \arrow[d, "\approx"] & H(\Delta(K)) \arrow[u, "\approx"'] \arrow[d, "\approx"] \\
H(|K|) & H(|K|) \arrow[l, "|\varphi'|_* = 1"'] \arrow[r, "|\varphi|_* = f_*"] & H(|K|)
\end{tikzcd}$$

from which it follows that

$$\lambda(f_*) = \lambda(|\varphi|_*(|\varphi'|_*)^{-1}) = \lambda(\varphi_*(\varphi'_*)^{-1})$$

By theorem 4.6.13, $(\varphi'_*)^{-1} = \tau_*$ and $\lambda(\varphi_*(\varphi'_*)^{-1}) = \lambda(\varphi_*\tau_*) = \lambda([C(\varphi)\tau]_*)$. Therefore $\lambda(f) = 0$. ∎

This yields the following generalization of the Brouwer fixed-point theorem.

8 COROLLARY *Every continuous map from a compact contractible polyhedron to itself has a fixed point.*

PROOF If X is contractible, $\tilde{H}(X) = 0$, and for any $f: X \to X$, $\lambda(f) = 1$ [because f_* is the identity map on $H_0(X) \approx \mathbf{Z}$]. ∎

This result is false for noncompact polyhedra. In fact, $\mathbf{R}$ is a contractible polyhedron and any translation different from $1_{\mathbf{R}}$ fails to have a fixed point.

Given a continuous map $f: S^n \to S^n$, the *degree of* f is the unique integer $\deg f$ such that

$$f_*(z) = (\deg f)z \qquad z \in \tilde{H}_n(S^n)$$

The following fact is obvious.

9 *For any map $f: S^n \to S^n$, $\lambda(f) = 1 + (-1)^n \deg f$.* ∎

Since the antipodal map $S^n \to S^n$ has no fixed points, the next result follows from theorem 7 and statement 9.

10 COROLLARY *The antipodal map of S^n has degree $(-1)^{n+1}$.* ∎

11 COROLLARY *If n is even, there is no continuous map $f: S^n \to S^n$ such that x and $f(x)$ are orthogonal for all $x \in S^n$.*

PROOF Assume that such a map exists. Then a homotopy $F: f \simeq 1_{S^n}$ is defined by

$$F(x,t) = \frac{(1 - t)f(x) + tx}{\|(1 - t)f(x) + tx\|}$$

This is well-defined, because the condition that x and $f(x)$ be orthogonal implies $\|(1 - t)f(x) + tx\|^2 = (1 - t)^2 + t^2 \neq 0$ for $0 \leq t \leq 1$. Since $f \simeq 1_{S^n}$, $\lambda(f) = \lambda(1_{S^n}) = 1 + (-1)^n \neq 0$. Hence, by theorem 7, f must have a fixed point, in contradiction to the orthogonality of x and $f(x)$ for all x. ∎

This last result is equivalent to the statement that an even-dimensional sphere S^n has no continuous tangent vector field which is nonzero everywhere on S^n. For odd n such vector fields do exist because the map $f: S^{2m-1} \to S^{2m-1}$ defined by

$$f(x_1, \ldots ,x_{2m}) = (-x_2, x_1, \ldots , -x_{2m}, x_{2m-1})$$

is continuous and has the property that x and $f(x)$ are orthogonal for all x.

Instead of considering vector fields, we consider one-parameter groups of homeomorphisms. A *flow* on X is a continuous map

$$\psi\colon \mathbf{R} \times X \to X$$

such that

$$\begin{array}{lll} (a) & \psi(t_1 + t_2, x) = \psi(t_1, \psi(t_2,x)) & t_1, t_2 \in \mathbf{R};\ x \in X \\ (b) & \psi(0,x) = x & x \in X \end{array}$$

For $t \in \mathbf{R}$ let $\psi_t\colon X \to X$ be defined by $\psi_t(x) = \psi(t,x)$. Then (a) and (b) imply $\psi_{-t} = (\psi_t)^{-1}$, and so ψ_t is a homeomorphism of X for all $t \in \mathbf{R}$. A *fixed point of the flow* is a point $x_0 \in X$ such that $\psi(t,x_0) = x_0$ for all $t \in \mathbf{R}$.

12 THEOREM *If X is a compact polyhedron with $\chi(X) \neq 0$, then any flow on X has a fixed point.*

PROOF Each ψ_t is homotopic to 1_X [by the homotopy $F\colon X \times I \to X$ defined by $F(x,t') = \psi((1 - t')t, x)$]. Therefore

$$\lambda(\psi_t) = \lambda(1_X) = \chi(X) \neq 0$$

Hence, by theorem 7, each ψ_t has fixed points. For $n \geq 1$ let A_n be the closed subset of X consisting of the fixed points of $\psi_{1/2^n}$. Then $A_{n+1} \subset A_n$, and $\{A_n\}$ is a decreasing sequence of nonempty closed subsets of the compact space X. Let $F = \bigcap A_n$. Then F is nonempty, and any point of F is fixed under ψ_t for all t of the form $1/2^n$ for $n \geq 1$. This implies that each point of F is fixed under ψ_t for all dyadic rationals $t = m/2^n$. Since the dyadic rationals are dense in $\mathbf{R}$, each point of F is fixed under ψ_t for all t. ∎

We now turn our attention to separation properties of the sphere.

13 LEMMA *If $A \subset S^n$ is homeomorphic to I^k for $0 \leq k \leq n$, then $\tilde{H}(S^n - A) = 0$.*

PROOF We prove this by induction on k. If $k = 0$, then A is a point and $S^n - A$ is homeomorphic to $\mathbf{R}^n$. Therefore $\tilde{H}(S^n - A) = 0$.

Assume the result for $k < m$, where $m \geq 1$, and let A be homeomorphic to I^m. Regard A as being homeomorphic to $B \times I$, where B is homeomorphic to I^{m-1}, by a homeomorphism $h\colon B \times I \to A$. Let $A' = h(B \times [0,\frac{1}{2}])$ and $A'' = h(B \times [\frac{1}{2},1])$. Then $A = A' \cup A''$ and $A' \cap A''$ is homeomorphic to $B \times \frac{1}{2}$. By the inductive assumption, $\tilde{H}(S^n - (A' \cap A'')) = 0$. Because $S^n - A'$ and $S^n - A''$ are open sets, they are excisive and from the exactness of the corresponding reduced Mayer-Vietoris sequence

$$i_*\colon \tilde{H}_q(S^n - A) \approx \tilde{H}_q(S^n - A') \oplus \tilde{H}_q(S^n - A'')$$

If $z \in \tilde{H}_q(S^n - A)$ is nonzero, then either $i'_* z \neq 0$ in $\tilde{H}_q(S^n - A')$ or $i''_* z \neq 0$ in $\tilde{H}_q(S^n - A'')$, where $i'\colon S^n - A \subset S^n - A'$ and $i''\colon S^n - A \subset S^n - A''$. Assume $i'_* z \neq 0$. We repeat the argument for A' and thus obtain a sequence of sets

$$A \supset A_1 \supset A_2 \supset \cdots$$

such that

(*a*) The inclusion $S^n - A \subset S^n - A_j$ maps z into a nonzero element of $\tilde{H}_q(S^n - A_j)$.
(*b*) $\cap A_i$ is homeomorphic to I^{m-1}

Because every compact subset of $S^n - \cap A_i$ is contained in $S^n - A_j$ for some j, it follows from theorem 4.4.6 that $\tilde{H}_q(S^n - \cap A_i) \approx \lim_{\rightarrow} \{\tilde{H}_q(S^n - A_j)\}$. This is a contradiction because, by condition (*a*), the element z determines a nonzero element of $\lim_{\rightarrow} \{\tilde{H}_q(S^n - A_j)\}$, but by condition (*b*) and the inductive assumption, $\tilde{H}_q(S^n - \cap A_i) = 0$. ∎

14 COROLLARY *Let B be a subset of S^n which is homeomorphic to S^k for $0 \le k \le n - 1$. Then*

$$\tilde{H}_q(S^n - B) \approx \begin{cases} 0 & q \neq n - k - 1 \\ \mathbf{Z} & q = n - k - 1 \end{cases}$$

PROOF We use induction on k. If $k = 0$, then B consists of two points and $S^n - B$ has the same homotopy type as S^{n-1}. Therefore

$$\tilde{H}_q(S^n - B) \approx \begin{cases} 0 & q \neq n - 1 \\ \mathbf{Z} & q = n - 1 \end{cases}$$

If $k \ge 1$, set $B = A_1 \cup A_2$, where A_1 and A_2 are closed hemispheres of S^k and assume the result valid for $k - 1$. Then A_1 and A_2 are homeomorphic to I^k and $A_1 \cap A_2$ is homeomorphic to S^{k-1}. Because $S^n - A_1$ and $S^n - A_2$ are open, $\{S^n - A_1, S^n - A_2\}$ is an excisive couple, and there is an exact reduced Mayer-Vietoris sequence

$$\to \tilde{H}_{q+1}(S^n - A_1) \oplus \tilde{H}_{q+1}(S^n - A_2) \to \tilde{H}_{q+1}(S^n - (A_1 \cap A_2)) \to \\ \tilde{H}_q(S^n - B) \to \tilde{H}_q(S^n - A_1) \oplus \tilde{H}_q(S^n - A_2) \to$$

By lemma 13, the groups at the ends vanish. The result then follows from the inductive assumption. ∎

For the special case of an $(n - 1)$-sphere imbedded in S^n, we obtain the following *Jordan-Brouwer separation theorem.*

15 THEOREM *An $(n - 1)$-sphere imbedded in S^n separates S^n into two components of which it is their common boundary.*

PROOF If $B \subset S^n$ is homeomorphic to S^{n-1}, then $\tilde{H}_0(S^n - B) \approx \mathbf{Z}$, by corollary 14. Therefore $S^n - B$ consists of two path components. Since $S^n - B$ is an open subset of S^n, it is locally path connected and its path components U and V, say, are its components.

Clearly, B contains the boundary of U and of V. To prove $B \subset \bar{U} \cap \bar{V}$, let $x \in B$ and let N be a neighborhood of x in S^n. Let $A \subset B \cap N$ be a subset such that $B - A$, is homeomorphic to I^{n-1}. Then $\tilde{H}(S^n - (B - A)) = 0$, by lemma 13, so $S^n - (B - A)$ is path connected. If $p \in U$ and $q \in V$, there is a path ω in $S^n - (B - A)$ from p to q. Because p and q are in different path components of $S^n - B$, ω meets A. Therefore A contains a point of $\bar{U}$ and a point of $\bar{V}$. Hence N meets $\bar{U}$ and $\bar{V}$, and $x \in \bar{U} \cap \bar{V}$. ∎

A related result is the following *Brouwer theorem on the invariance of domain.*

16 THEOREM *If U and V are homeomorphic subsets of S^n and U is open in S^n, then V is open in S^n.*

PROOF Let h: $U \to V$ be a homeomorphism and let $h(x) = y$. Let A be a neighborhood of x in U that is homeomorphic to I^n and with boundary B homeomorphic to S^{n-1}. Let $A' = h(A) \subset V$ and let $B' = h(B)$. By lemma 13, $S^n - A'$ is connected, and by theorem 15, $S^n - B'$ has two components. Because

$$S^n - B' = (S^n - A') \cup (A' - B')$$

and $S^n - A'$ and $A' - B'$ are connected, they are the components of $S^n - B'$. Therefore $A' - B'$ is an open subset of S^n. Since $y \in A' - B' \subset V$ and y was arbitrary, V is open in S^n. ∎

8 AXIOMATIC CHARACTERIZATION OF HOMOLOGY

A simple set of axioms characterizing homology on the class of compact polyhedral pairs has been given by Eilenberg and Steenrod[1]. This section describes the axiom system and related concepts. For compact polyhedral pairs, the axioms are categorical (that is, two theories satisfying them are isomorphic). Thus the axioms are basic theorems from which other properties of homology theories can be deduced. In many cases, proofs based on the axioms are simpler and more elegant than proofs which refer back to the original definition of the homology theory.

To formulate the axioms it is usual to start with a suitable category of topological pairs and maps (called "admissible categories" by Eilenberg and Steenrod). We shall not define these categories. The category of all topological pairs is such a category, and so are its full subcategories defined by the polyhedral pairs and defined by the compact polyhedral pairs. For our purposes we shall always regard a homology theory as defined on the category of all topological pairs, and we identify a space X with the pair $(X, \varnothing)$.

A *homology theory* H and ∂ consists of

(*a*) A covariant functor H from the category of topological pairs and maps to the category of graded abelian groups and homomorphisms of degree 0 [that is, $H(X,A) = \{H_q(X,A)\}$]

[1] See S. Eilenberg and N. E. Steenrod, "Foundations of Algebraic Topology," Princeton University Press, Princeton, N.J., 1952.

(*b*) A natural transformation ∂ of degree -1 from the functor H on (X,A) to the functor H on $(A,\varnothing)$ [that is, $\partial(X,A) = \{\partial_q(X,A)\colon H_q(X,A) \to H_{q-1}(A)\}$].

These satisfy the following axioms.

1 HOMOTOPY AXIOM *If* $f_0, f_1\colon (X,A) \to (Y,B)$ *are homotopic, then*

$$H(f_0) = H(f_1)\colon H(X,A) \to H(Y,B)$$

2 EXACTNESS AXIOM *For any pair* (X,A) *with inclusion maps* $i\colon A \subset X$ *and* $j\colon X \subset (X,A)$ *there is an exact sequence*

$$\cdots \xrightarrow{\partial_{q+1}(X,A)} H_q(A) \xrightarrow{H_q(i)} H_q(X) \xrightarrow{H_q(j)} H_q(X,A) \xrightarrow{\partial_q(X,A)} H_{q-1}(A) \xrightarrow{H_{q-1}(i)} \cdots$$

3 EXCISION AXIOM *For any pair* (X,A), *if* U *is an open subset of* X *such that* $\bar{U} \subset \operatorname{int} A$, *then the excision map* $j\colon (X - U, A - U) \subset (X,A)$ *induces an isomorphism*

$$H(j)\colon H(X - U, A - U) \approx H(X,A)$$

4 DIMENSION AXIOM *On the full subcategory of one-point spaces, there is a natural equivalence of* H *with the constant functor; that is, if* P *is a one-point space, then*

$$H_q(P) \approx \begin{cases} 0 & q \neq 0 \\ \mathbf{Z} & q = 0 \end{cases}$$

Obviously, the homotopy axiom is equivalent to the condition that the homology theory can be factored through the homotopy category of topological pairs.

Singular homology theory is an example of a homology theory. In fact, the homotopy axiom is a consequence of theorem 4.4.9, the exactness axiom is a consequence of theorem 4.5.4, the excision axiom is a consequence of corollary 4.6.5, and the dimension axiom is a consequence of lemmas 4.4.1 and 4.3.1. Therefore, there exist homology theories.

Corresponding to any homology theory there are reduced groups defined as follows. If X is a nonempty space, let $c\colon X \to P$ be the unique map from X to some one-point space P. The reduced group $\tilde{H}(X)$ is defined to be the kernel of the homomorphism

$$H(c)\colon H(X) \to H(P)$$

Because c has a right inverse, so does $H(c)$. Therefore

$$H(X) \approx \tilde{H}(X) \oplus H(P)$$

and the reduced groups have properties similar to those of the reduced singular groups.

Given a triple $B \subset A \subset X$, let $k\colon A \subset (A,B)$ and define $\partial(X,A,B)\colon H(X,A) \to H(A,B)$ to be the composite

$$\partial(X,A,B) = H(k)\partial(X,A)\colon H(X,A) \to H(A) \to H(A,B)$$

5 THEOREM *For any triple (X,A,B), with inclusion maps $i\colon (A,B) \subset (X,B)$ and $j\colon (X,B) \subset (X,A)$, there is an exact sequence*

$$\cdots \to H_q(A,B) \xrightarrow{H_q(i)} H_q(X,B) \xrightarrow{H_q(j)} H_q(X,A) \xrightarrow{\partial_q(X,A,B)} H_{q-1}(A,B) \to \cdots$$

PROOF The proof involves diagram chasing based on the exactness axiom 2. We prove exactness at $H_q(A,B)$ and leave the other parts of the proof to the reader.

(*a*) im $\partial_{q+1}(X,A,B) \subset \ker H_q(i)$. $H_q(i)\partial_{q+1}(X,A,B)$ is the composite

$$H_{q+1}(X,A) \xrightarrow{\partial_{q+1}(X,A)} H_q(A) \xrightarrow{H_q(k)} H_q(A,B) \xrightarrow{H_q(i)} H_q(X,B)$$

which also equals the composite

$$H_{q+1}(X,A) \xrightarrow{\partial_{q+1}(X,A)} H_q(A) \xrightarrow{H_q(i')} H_q(X) \xrightarrow{H_q(i'')} H_q(X,B)$$

where $i'\colon A \subset X$ and $i''\colon X \subset (X,B)$. By axiom 2, $H_q(i')\partial_{q+1}(X,A) = 0$. Therefore $H_q(i)\partial_{q+1}(X,A,B) = 0$.

(*b*) $\ker H_q(i) \subset$ im $\partial_{q+1}(X,A,B)$. Let $z \in H_q(A,B)$ be such that $H_q(i)z = 0$. Then $\partial_q(X,B)H_q(i)z = 0$, and because $\partial_q(A,B) = \partial_q(X,B)H_q(i)$, $\partial_q(A,B)z = 0$. By axiom 2, there is $z' \in H_q(A)$ such that $H_q(k)z' = z$. Because the composite

$$H_q(A) \xrightarrow{H_q(i')} H_q(X) \xrightarrow{H_q(i'')} H_q(X,B)$$

equals the composite $H_q(i)H_q(k)$, it follows that

$$H_q(i'')H_q(i')z' = H_q(i)H_q(k)z' = H_q(i)z = 0$$

By axiom 2, there is $z'' \in H_q(B)$ such that if $j'\colon B \subset X$, then $H_q(i')z' = H_q(j')z''$. Given $j''\colon B \subset A$, then $H_q(j') = H_q(i')H_q(j'')$. Therefore $H_q(i')(z' - H_q(j'')z'') = 0$. Again by axiom 2, there is $\bar{z} \in H_{q+1}(X,A)$ such that $\partial_{q+1}(X,A)\bar{z} = z' - H_q(j'')z''$. Then, because $H_q(k)H_q(j'') = 0$,

$$\partial_{q+1}(X,A,B)\bar{z} = H_q(k)\partial_{q+1}(X,A)\bar{z} = H_q(k)z' - H_q(k)H_q(j'')z'' = z$$

which shows that z is in im $\partial_{q+1}(X,A,B)$. ∎

The exact sequence of theorem 5 is called the *homology sequence of the triple* (X,A,B). If $B = \varnothing$, it reduces to the homology sequence of the pair (X,A).

Let H and ∂ and H' and ∂' be homology theories. A *homomorphism* from H and ∂ to H' and ∂' is a natural transformation h from H to H' commuting with ∂ and ∂'. That is, for every (X,A) there is a commutative diagram

$$\begin{array}{ccc} H(X,A) & \xrightarrow{\partial(X,A)} & H(A) \\ {\scriptstyle h(X,A)}\downarrow & & \downarrow{\scriptstyle h(A)} \\ H'(X,A) & \xrightarrow{\partial'(X,A)} & H'(A) \end{array}$$

in which the vertical maps are homomorphisms of degree 0. In view of the dimension axiom, a homomorphism h induces a homomorphism $h_0\colon \mathbf{Z} \to \mathbf{Z}$

that characterizes h on one-point spaces. The main result proved by Eilenberg and Steenrod is that corresponding to any homomorphism h_0: $\mathbf{Z} \to \mathbf{Z}$ there exists a unique homomorphism h from H and ∂ to H' and ∂', on the category of compact polyhedral pairs, which induces h_0. We shall not prove this, but shall content ourselves with proving that a homomorphism h which is an isomorphism for one-point spaces is an isomorphism for any compact polyhedral pair. This will illustrate how the axioms can be used and will suffice for our later applications.

The following is an easy consequence of the exactness axiom and the five lemma (or of theorem 5 and axiom 2).

6 LEMMA *Let $A' \subset A \subset X$. Then $H(A') \approx H(A)$ if and only if $H(X,A') \approx H(X,A)$ (both maps induced by inclusion).* ∎

We now prove a stronger excision property. A map f: $(X,A) \to (Y,B)$ is called a *relative homeomorphism* if f maps $X - A$ homeomorphically onto $Y - B$. Following are some examples.

7 An excision map $(X - U, A - U) \subset (X,A)$, where $U \subset A$, is a relative homeomorphism.

8 If X is obtained from A by adjoining an n-cell e and f: $(E^n,S^{n-1}) \to (e,\dot{e})$ is a characteristic map for e, then f is a relative homeomorphism.

9 THEOREM *Let X be a compact Hausdorff space and let A be a closed subset of X which is a strong deformation retract of one of its closed neighborhoods in X. Let f: $(X,A) \to (Y,B)$ be a relative homeomorphism, where Y is a Hausdorff space and B is closed in Y. Then, for any homology theory $H(f)$: $H(X,A) \approx H(Y,B)$.*

PROOF Let N be a closed neighborhood of A in X such that A is a strong deformation retract of N and let U be an open subset of X such that $A \subset U \subset \bar{U} \subset N$ (U exists because X is a normal space). Let F: $N \times I \to N$ be a strong deformation retraction of N to A.

Define $N' = f(N) \cup B$, $U' = f(U) \cup B$, and F': $N' \times I \to N'$ by

$$\begin{aligned} F'(y,t) &= y && y \in B, t \in I \\ F'(y,t) &= fF(f^{-1}(y),t) && y \in f(N), t \in I \end{aligned}$$

Then F' is well-defined and continuous on each of the closed sets $B \times I$ and $f(N) \times I$. Therefore F' is continuous and is easily verified to be a strong deformation retraction of N' to B. Because $X - \bar{U}$ is open in $X - A$, $Y - (f(\bar{U}) \cup B)$ is open in $Y - B$, and because B is closed, it is open in Y. Therefore $f(\bar{U}) \cup B$ is closed in Y, and $\bar{U}' \subset f(\bar{U}) \cup B \subset N'$. Because $X - U$ is a closed, and hence compact, subset of X, $f(X - U) = Y - U'$ is a compact subset of Y. Because Y is a Hausdorff space, $Y - U'$ is closed in Y, and U' is open in Y. We have $B \subset U' \subset \bar{U}' \subset N'$ and a commutative diagram

$$\begin{array}{ccccc}
H(X,A) & \xrightarrow{\approx} & H(X,N) & \xleftarrow{\approx} & H(X-U, N-U) \\
{\scriptstyle H(f)}\downarrow & & \downarrow & & \downarrow\approx \\
H(Y,B) & \xrightarrow{\approx} & H(Y,N') & \xleftarrow{\approx} & H(Y-U', N'-U')
\end{array}$$

where the vertical maps are induced by f and the horizontal maps are induced by inclusion maps. Because A and B are deformation retracts of N and N', respectively, $H(A) \approx H(N)$ and $H(B) \approx H(N')$. It follows from lemma 6 that the left-hand horizontal maps are isomorphisms. The right-hand horizontal maps are isomorphisms by the excision axiom. The right-hand vertical map is an isomorphism because it is induced by a homeomorphism. From the commutativity of the diagram, it follows that $H(f)$ is an isomorphism. ■

10 THEOREM *Let h be a homomorphism from H and ∂ to H' and ∂' which is an isomorphism for one-point spaces. Then, for any compact polyhedral pair (X,A), $h(X,A)$: $H(X,A) \approx H'(X,A)$.*

PROOF By the five lemma, it suffices to prove $h(X)$: $H(X) \approx H'(X)$ for any compact polyhedron X. Hence, let K be a finite simplicial complex. We need only prove that $h(|K|)$: $H(|K|) \approx H'(|K|)$. We prove this by induction on the number of simplexes of K. If K has just one simplex, $|K|$ is a one-point space, and $h(|K|)$ is an isomorphism by hypothesis.

Assume that K has m simplexes, where $m > 0$, and that h is an isomorphism for the space of any simplicial complex with fewer than m simplexes. Assume $\dim K = n$ and let s be an n-simplex of K. Let L be the subcomplex consisting of all simplexes of K different from s. By the five lemma and the exactness axiom, $h(|K|)$ is an isomorphism if and only if $h(|K|,|L|)$ is an isomorphism. If j: $(|s|,|\dot{s}|) \subset (|K|,|L|)$, it follows from theorem 9 that $H(j)$ and $H'(j)$ are isomorphisms. Hence we need only prove that $h(|s|,|\dot{s}|)$ is an isomorphism.

If $n = 0$, $(|s|,|\dot{s}|)$ is a one-point space, and $h(|s|,|\dot{s}|)$ is an isomorphism by hypothesis. If $n > 0$, because $|s|$ has the same homotopy type as a one-point space, $h(|s|)$ is an isomorphism. By the five lemma and the exactness axiom, $h(|s|,|\dot{s}|)$ is an isomorphism if and only if $h(|\dot{s}|)$ is an isomorphism. Because $\dot{s}$ is a proper subcomplex of K, $h(|\dot{s}|)$ is an isomorphism by the inductive hypothesis. ■

To extend this result to arbitrary polyhedral pairs (not merely compact ones), we add an additional axiom. A pair (X,A) with X compact and A closed in X is called a *compact pair.*

11 AXIOM OF COMPACT SUPPORTS *Given any pair (X,A) and given $z \in H_q(X,A)$, there is a compact pair $(X',A') \subset (X,A)$ such that z is in the image of $H(X',A') \to H(X,A)$.*

A homology theory H and ∂ satisfying axiom 11 is called a *homology theory with compact supports* (Eilenberg and Steenrod use the term "homology theory with compact carriers"). It is clear that singular homology theory is a homology theory with compact supports. We shall see that any homology theory with compact supports satisfies the analogue of theorem 4.4.6. The following lemma is the main point in proving this.

12 LEMMA *Let H be a homology theory with compact supports and let (X',A') be a compact pair in (X,A). Given $z \in H_q(X',A')$ in the kernel of $H_q(X',A') \to H_q(X,A)$, there is a compact pair (X'',A''), with $(X',A') \subset (X'',A'') \subset (X,A)$, such that z is in the kernel of $H(X',A') \to H(X'',A'')$.*

PROOF In the proof all unlabeled maps are induced by inclusion. z is in the kernel of the composite

$$H_q(X',A') \xrightarrow{H_q(i)} H_q(X,A') \to H_q(X,A)$$

By theorem 5, $H_q(i)z$ is in the image of $H_q(A,A') \to H_q(X,A')$. By axiom 11, there is a compact space A'' such that $A' \subset A'' \subset A$ and such that $H_q(i)z$ is in the image of the composite $H_q(A'',A') \to H_q(A,A') \to H_q(X,A')$. By theorem 5, the composite $H_q(A'',A') \to H_q(X,A') \to H_q(X,A'')$ is trivial. Therefore z is in the kernel of $H_q(X',A') \to H_q(X,A'')$ for some compact A'' containing A'.

Because z is in the kernel of the composite

$$H_q(X',A') \xrightarrow{H_q(j)} H_q(X' \cup A'', A'') \to H_q(X,A'')$$

it follows from theorem 5, that $H_q(j)z$ is in the image of

$$\partial_{q+1}\colon H_{q+1}(X, X' \cup A'') \to H_q(X' \cup A'', A'')$$

By axiom 11, there is a compact X'' containing $X' \cup A''$ such that $H_q(j)z$ is in the image of the composite

$$H_{q+1}(X'', X' \cup A'') \to H_{q+1}(X, X' \cup A'') \xrightarrow{\partial_{q+1}} H_q(X' \cup A'', A'')$$

This composite is also equal to the map $\partial_{q+1}\colon H_{q+1}(X'', X' \cup A'') \to H_q(X' \cup A'', A'')$. By theorem 5, the composite

$$H_{q+1}(X'', X' \cup A'') \xrightarrow{\partial_{q+1}} H_q(X' \cup A'', A'') \to H_q(X'',A'')$$

is trivial. Therefore, z is in the kernel of $H_q(X',A') \to H_q(X'',A'')$. ∎

For any pair (X,A) the family of compact pairs (X',A') contained in (X,A) is directed by inclusion. For any homology theory H and ∂ the groups $\{H(X',A') \mid (X',A') \text{ compact} \subset (X,A)\}$ constitute a direct system, and the maps $H(X',A') \to H(X,A)$ define a homomorphism $i\colon \lim_{\to} \{H(X',A')\} \to H(X,A)$.

13 THEOREM *A homology theory H and ∂ has compact supports if and only if for any pair (X,A), $i\colon \lim_{\to} \{H(X',A')\} \approx H(X,A)$, where (X',A') varies over the family of compact pairs contained in (X,A).*

PROOF It is clear that axiom 11 is equivalent to the condition that i be an epimorphism. Hence, if i is an isomorphism, H and ∂ has compact supports. Conversely, if H has compact supports, i is an epimorphism, and lemma 12 implies that i is also a monomorphism. ∎

14 THEOREM *Let h be a homomorphism from H and ∂ to H' and ∂' that is an isomorphism for one-point spaces. If H and ∂ and H' and ∂' have compact supports, h is an isomorphism for any polyhedral pair.*

PROOF This follows from theorems 10 and 13 and from the fact that for any polyhedral pair (X,A) the compact polyhedral pairs (X',A') contained in it are cofinal in the family of all compact pairs in (X,A). ∎

EXERCISES

A CHAIN HOMOTOPY CLASSES

1 For chain complexes C and C' show that $[C;C']$ is an abelian group (with group operation $[\tau_1] + [\tau_2] = [\tau_1 + \tau_2]$) and that there is a homomorphism

$$\varphi: [C;C'] \to \operatorname{Hom}(H(C),H(C'))$$

such that $\varphi[\tau] = \tau_*$.

2 If C is a free chain complex, prove that the homomorphism φ is an epimorphism.

3 If C is a free chain complex and $H(C)$ is also free, prove that φ is an isomorphism.

B EULER CHARACTERISTICS

1 Let (X,A) be a pair and assume that two of the three graded groups $H(A)$, $H(X)$, and $H(X,A)$ are finitely generated. Prove that the third is also finitely generated and that $\chi(X) = \chi(A) + \chi(X,A)$.

2 Let $\{X_1,X_2\}$ be an excisive couple of subsets of X such that $H(X_1)$ and $H(X_2)$ are finitely generated. Prove that $H(X_1 \cup X_2)$ is finitely generated if and only if $H(X_1 \cap X_2)$ is finitely generated, in which case

$$\chi(X_1) + \chi(X_2) = \chi(X_1 \cup X_2) + \chi(X_1 \cap X_2)$$

3 Let γ be an integer-valued function defined on the class of compact polyhedra with base points such that

(*a*) If (X,x_0) is homeomorphic to (Y,y_0), then $\gamma(X,x_0) = \gamma(Y,y_0)$.
(*b*) If (X,A) is a compact polyhedral pair and $x_0 \in A$, then $\gamma(X,x_0) = \gamma(A,x_0) + \gamma(X/A,x_0')$, where X/A denotes the space obtained by collapsing A to a single point x_0'.

Prove that for any X

$$\gamma(X,x_0) = \gamma(S^0,p_0)\chi(X,x_0)$$

[*Hint:*[1] Prove first that if z_0 is a base point of E^n in S^{n-1}, then $\gamma(E^n,z_0) = 0$. Show that the result is true for $X = S^n$, and then use induction on the number of simplexes in a triangulation of X.]

4 If X and Y are compact polyhedra, prove that

$$\chi(X \times Y) = \chi(X)\chi(Y)$$

[1] See C. E. Watts, On the Euler characteristic of polyhedra, *Proceedings of the American Mathematical Society*, vol. 13, pp. 304–306, 1962.

C EXAMPLES

1 Let s be an n-simplex and let $(s)^m$ be its m-dimensional skeleton. Compute $H((s)^m)$.

2 Compute the homology group of an arbitrary surface.

3 Compute the homology group of the lens space $L(p,q)$.

4 Let A be a subspace of S^n which is homeomorphic to the one-point union $S^p \vee S^q$. Compute $H(S^n - A)$.

5 Let X be the space obtained from a closed triangle with vertices v_0, v_1, and v_2 by identifying the edges v_0v_1, v_1v_2, and v_2v_0 linearly with the edges v_1v_2, v_2v_0, and v_0v_1, respectively. Compute $H(X)$.

6 Given an integer $n > 0$ and an integer $m > 1$, prove that there exists a compact polyhedron X such that

$$\tilde{H}_q(X) = \begin{cases} 0 & q \neq n \\ \mathbf{Z}_m & q = n \end{cases}$$

7 Let H be a finitely generated nonnegative graded abelian group such that H_0 is a free abelian group. Prove that there exists a compact polyhedron X such that $\tilde{H}(X) \approx H$.

D JOINS AND PRODUCTS

1 Prove that for any space X there are isomorphisms

$$\tilde{H}_q(X) \approx \tilde{H}_{q+1}(X * S^0)$$

(*Hint:* If Y is contractible, so is $X * Y$.)

2 Prove that for any space X there are isomorphisms

$$H_q(X \times S^n, X \times p_0) \approx H_{q-n}(X)$$

[*Hint:* Use induction on n and the fact that if Y is contractible, $H(X \times Y, X \times y_0) = 0$.]

3 Compute the homology group of the n-dimensional torus $(S^1)^n$.

4 If a space is homeomorphic to a finite product of spheres, prove that the set of spheres which are the factors is unique.

E ORIENTATION

1 Let K be an n-dimensional pseudomanifold. Prove that it is possible to enumerate the n-simplexes of K in a (finite or infinite) sequence $s_0, s_1, \ldots, s_q, \ldots$ and to find a sequence $s_1', s_2', \ldots, s_q', \ldots$ of $(n-1)$-simplexes of K such that for $q \geq 1$, s_q' is a face of s_q and also a face of s_i for some $i < q$.

2 If K is a finite n-dimensional pseudomanifold, prove that exactly one of the following holds:

(*a*) $H_n(K,\dot{K}) \approx \mathbf{Z}$ and $H_{n-1}(K,\dot{K})$ has no torsion.
(*b*) $H_n(K,\dot{K}) = 0$ and $H_{n-1}(K,\dot{K})$ has torsion subgroup isomorphic to $\mathbf{Z}_2$.

3 Let K be a finite simplicial complex which is homogeneously n-dimensional and such that every $(n-1)$-simplex of K is the face of at most two n-simplexes of K. Let $\dot{K}$ be the subcomplex of K generated by the $(n-1)$-simplexes of K which are faces of exactly one n-simplex of K. Prove that if $(K,\dot{K})$ satisfies either (*a*) or (*b*) of exercise 2 above, then K is an n-dimensional pseudomanifold.

A finite n-dimensional pseudomanifold is said to be *orientable* (or *nonorientable*) if it satisfies (*a*) (or (*b*)) of exercise 2. An *orientation* of an orientable n-dimensional pseudomanifold K is a generator of $H_n(K,\dot{K})$, and an *oriented* n-dimensional pseudomanifold is an n-dimensional pseudomanifold together with an orientation of it.

4 Let $z \in H_n(K,\dot{K})$ be an orientation of a finite n-dimensional pseudomanifold. If s is any n-simplex of K, prove that there is a unique orientation of s, denoted by $z \mid s \in H_n(s,\dot{s})$ and called the *induced orientation of* s, characterized by the property that z and $z \mid s$ correspond under the homomorphisms

$$H_n(K,\dot{K}) \to H_n(K, K - s) \xleftarrow{\approx} H_n(s,\dot{s})$$

A collection of orientations $\{\sigma(s) \in H_n(s,\dot{s})\}$ for each n-simplex s of an n-dimensional pseudomanifold is called *compatible* if for any $(n-1)$-simplex s' of $K - \dot{K}$ which is a face of the two n-simplexes s_1 and s_2 of K, $\sigma(s_1)$ and $-\sigma(s_2)$ correspond under the homomorphisms

$$\begin{tikzcd}
H_n(s_1,\dot{s}_1) \arrow[r, "\partial"] & H_{n-1}(\dot{s}_1) \arrow[r] & H_{n-1}(\dot{s}_1, \dot{s}_1 - s') & \\
 & & & H_{n-1}(s',\dot{s}') \arrow[ul, "\approx"'] \arrow[dl, "\approx"] \\
H_n(s_2,\dot{s}_2) \arrow[r, "\partial"] & H_{n-1}(\dot{s}_2) \arrow[r] & H_{n-1}(\dot{s}_2, \dot{s}_2 - s') &
\end{tikzcd}$$

5 If z is an orientation of a finite n-dimensional pseudomanifold, prove that the collection $\{z \mid s\}$ is compatible. Conversely, given a compatible collection $\{\sigma(s)\}$ of orientations of the n-simplexes s of a finite n-dimensional pseudomanifold K, prove that there is a unique orientation z of K such that $z \mid s = \sigma(s)$ for each n-simplex s of K. Use this to define orientability for arbitrary (nonfinite) n-dimensional pseudomanifolds. [*Hint:* Identify $H_n(K,K^{n-1})$ with indexed collections $\{\sigma(s) \in H_n(s,\dot{s})\}$, where s varies over the n-simplexes of K, and show that the image of the homomorphism $H_n(K,\dot{K}) \to H_n(K,K^{n-1})$ consists of the compatible collections.]

F DEGREES OF MAPS

Let K_1 and K_2 be finite n-dimensional pseudomanifolds with orientations z_1 and z_2, respectively. Given a continuous map $f: (|K_1|,|\dot{K}_1|) \to (|K_2|,|\dot{K}_2|)$, its *degree*, denoted by $\deg f$, is the unique integer such that $f_*(z_1) = (\deg f)z_2$ [where we regard $z_1 \in H_n(|K_1|,|\dot{K}_1|)$ and $z_2 \in H_n(|K_2|,|\dot{K}_2|)$].

1 Let $\varphi: (K_1,\dot{K}_1) \to (K_2,\dot{K}_2)$ be a simplicial approximation to f, let s_2 be a fixed n-simplex of K_2, and let $m_+(\varphi)$ (or $m_-(\varphi)$) be the number of n-simplexes s_1 of K_1 such that φ maps the induced orientation $z_1 \mid s_1$ into the induced orientation $z_2 \mid s_2$ (or into $-z_2 \mid s_2$). Prove that $\deg f = m_+(\varphi) - m_-(\varphi)$.

2 In case K is a finite orientable n-dimensional pseudomanifold and $f: (|K|,|\dot{K}|) \to (|K|,|\dot{K}|)$, there is a unique integer $\deg f$ such that $f_*(z) = (\deg f)z$ for any $z \in H_n(|K|,|\dot{K}|)$. Prove that if $f, g: (|K|,|\dot{K}|) \to (|K|,|\dot{K}|)$, then $\deg (g \circ f) = (\deg g)(\deg f)$.

3 Let $f: S^n \to S^n$ be a map such that $f(E^n_+) \subset E^n_+$, $f(E^n_-) \subset E^n_-$ and let $f': S^{n-1} \to S^{n-1}$ be the map defined by f. Prove that $\deg f = \deg f'$.

4 Show that for any $n \geq 1$ and any integer m there is a map $f: S^n \to S^n$ such that $\deg f = m$.

G TOPOLOGICAL INVARIANCE OF PSEUDOMANIFOLDS

1 Let K be a simplicial complex and let $x \in \langle s \rangle$, where s is a simplex of K. Prove that there is an isomorphism

$$H(|K|, |K| - \text{st } s) \approx H(|K|, |K| - x)$$

2 Let K be a simplicial complex and let $x \in \langle s \rangle$, where s is a *principal* n-simplex of K (that is, s is not a proper face of any simplex of K). Prove that

$$H_q(|K|, |K| - x) \approx \begin{cases} 0 & q \neq n \\ \mathbf{Z} & q = n \end{cases}$$

3 Prove that a locally compact polyhedron X has dimension n if and only if n is the largest integer such that there exist points $x \in X$, with $H_n(X, X - x) \neq 0$.

4 Let X be a finite dimensional polyhedron and for each n let X_n be the closure of the set of all $x \in X$ having a neighborhood U such that $H_n(X, X - y) \approx \mathbf{Z}$ for all $y \in U$. If K is any simplicial complex triangulating X and K_n is the subcomplex of K generated by the principal n-simplexes of K, prove that K_n triangulates X_n.

5 Prove that the property of being homogeneously n-dimensional is a topologically invariant property of simplicial complexes (and so we can speak of a homogeneously n-dimensional polyhedron).

6 Let K be an arbitrary simplicial complex triangulating a homogeneously n-dimensional polyhedron X. Prove that every $(n - 1)$-simplex of K is the face of at most two n-simplexes of K if and only if $H_q(A, A - x) = 0$ for all $x \in A$ and all $q \geq n - 1$, where A is the closure in X of the set $\{x \in X \mid H_n(X, X - x) \text{ is noncyclic}\}$.

7 Let X be a homogeneously n-dimensional polyhedron satisfying exercise 6 and let $\dot{X} = B_{n-1}$, where B is the closure in X of the set $\{x \in X \mid H_n(X, X - x) = 0\}$ and where B_{n-1} is defined in terms of B, as in exercise 4. If K is any simplicial complex triangulating X, prove that the subcomplex of K generated by the $(n - 1)$-simplexes of K which are faces of exactly one n-simplex of K triangulates $\dot{X}$.

8 Prove that the property of being a finite n-dimensional pseudomanifold is a topologically invariant property of simplicial complexes.

H EDGE-PATH GROUPS

1 Let K be a connected simplicial complex with a base vertex $v_0 \in K$. Given an edge $e = (v_0, v_1)$, of K, let $[e]$ be the oriented 1-simplex $[v_0, v_1]$. If $\zeta = e_1 e_2 \cdots e_r$ is a closed edge path of K at v_0, let $\psi(\zeta) = [e_1] + [e_2] + \cdots + [e_r] \in C_1(K)$. Prove that $\psi(\zeta)$ is a cycle and that if ζ and ζ' are equivalent edge paths, then $\psi(\zeta)$ and $\psi(\zeta')$ are homologous.

2 Prove that there is a natural transformation ψ: $E(K,v_0) \to H_1(K)$ (on the category of connected simplicial complexes with a base vertex) defined by $\psi[\zeta] = \{\psi(\zeta)\}$.

3 Prove that the homomorphism ψ is an epimorphism and has kernel equal to the commutator subgroup of $E(K,v_0)$.

I AXIOMATIC HOMOLOGY THEORY

In this group of exercises H will denote an arbitrary homology theory.

1 Let X_1 and X_2 be subspaces of a space X. Prove that the following are equivalent:

(*a*) The excision map $(X_1, X_1 \cap X_2) \subset (X_1 \cup X_2, X_2)$ induces an isomorphism of homology.

(*b*) The excision map $(X_2, X_1 \cap X_2) \subset (X_1 \cup X_2, X_1)$ induces an isomorphism of homology.

(*c*) The inclusion maps

$$i_1\colon (X_1, X_1 \cap X_2) \subset (X_1 \cup X_2, X_1 \cap X_2)$$

and

$$i_2\colon (X_2, X_1 \cap X_2) \subset (X_1 \cup X_2, X_1 \cap X_2)$$

induce monomorphisms on homology and

$$H(X_1 \cup X_2, X_1 \cap X_2) \approx i_{1*}H(X_1, X_1 \cap X_2) \oplus i_{2*}H(X_2, X_1 \cap X_2)$$

(*d*) The inclusion maps

$$j_1\colon (X_1 \cup X_2, X_1 \cap X_2) \subset (X_1 \cup X_2, X_1)$$

and

$$j_2\colon (X_1 \cup X_2, X_1 \cap X_2) \subset (X_1 \cup X_2, X_2)$$

induce epimorphisms on homology and j_{1*} and j_{2*} induce an isomorphism

$$H(X_1 \cup X_2, X_1 \cap X_2) \approx H(X_1 \cup X_2, X_1) \oplus H(X_1 \cup X_2, X_2)$$

(*e*) For any $A \subset X_1 \cap X_2$ there is an exact Mayer-Vietoris sequence

$$\cdots \to H_q(X_1 \cap X_2, A) \to H_q(X_1,A) \oplus H_q(X_2,A) \to H_q(X_1 \cup X_2, A) \to H_{q-1}(X_1 \cap X_2, A) \to \cdots$$

(*f*) For any $Y \supset X_1 \cup X_2$ there is an exact Mayer-Vietoris sequence

$$\cdots \to H_q(Y, X_1 \cap X_2) \to H_q(Y,X_1) \oplus H_q(Y,X_2) \to H_q(Y, X_1 \cup X_2) \to H_{q-1}(Y, X_1 \cap X_2) \to \cdots$$

2 Let $X_1, \ldots, X_m$ and A be closed subspaces of a space X such that

(*a*) $X = \cup X_i$.
(*b*) $X_i \cap X_j = A$ if $i \neq j$.
(*c*) $\overline{X_i - A}$ is disjoint from $\overline{X_j - A}$ if $i \neq j$.

Prove that the homomorphisms $H(X_i,A) \to H(X,A)$ are monomorphisms and $H(X,A)$ is isomorphic to the direct sum of the images.

3 Let $\{X_j\}_{j \in J}$ (with J possibly infinite) be a collection of closed subsets of a space X and let A be a subspace of X such that (*a*), (*b*), and (*c*) of exercise 2 above are satisfied. Assume also that every compact subset of X is contained in a finite union of $\{X_j\}$ and that H is a homology theory with compact supports. Prove that $H(X,A) \approx \bigoplus_{j \in J} H(X_j,A)$.

4 Let (X,A) be a topological pair and let $\{X_s\}$ be a family of subspaces of X indexed by the integers such that

(*a*) $A = X_{-1}$.
(*b*) $X_s \subset X_{s+1}$ for all s.
(*c*) $X = \cup X_s$ and every compact subset of X is contained in X_s for some s.
(*d*) $H_q(X_s,X_{s-1}) = 0$ if $q \neq s$ and $s \geq 0$.

Let $C = \{C_q,\partial_q\}$ be the nonnegative chain complex with $C_q = H_q(X_q,X_{q-1})$ for $q \geq 0$ and ∂_q the connecting homomorphism of the triple (X_q, X_{q-1}, X_{q-2}) for $q \geq 1$. If H has compact supports, prove that $H(X,A) \approx H(C)$. [*Hint:* Prove that there are exact sequences

$$H_{q+1}(X_{q+1},X_q) \xrightarrow{\partial} H_q(X_q,A) \to H_q(X,A) \to 0$$

and

$$0 \to H_q(X_q,A) \to H_q(X_q,X_{q-1}) \to H_{q-1}(X_{q-1},A)]$$

5 Let H be a homology theory defined on the category of compact pairs. Prove that there is an extension of H to a homology theory $\bar{H}$ with compact supports such that $\bar{H}(X,A) = \lim_{\to} \{H(X',A') \mid (X',A') \text{ a compact pair in } (X,A)\}$.

CHAPTER FIVE
PRODUCTS

WE ARE NOW READY TO EXTEND THE DEFINITION OF HOMOLOGY TO MORE GENERAL coefficients. In this framework the homology considered in the last chapter appears as the special case of integral coefficients. The extension is done in a purely algebraic way. Given a chain complex C and an abelian group G, their tensor product is the chain complex $C \otimes G = \{C_q \otimes G, \partial_q \otimes 1\}$, and the homology of $C \otimes G$ is defined to be the homology of C, with coefficients G.

We shall also introduce the concepts of cochain complex and cohomology. These are dual to the concepts of chain complex and homology and arise on replacing the tensor-product functor by the functor Hom.

We shall establish universal-coefficient formulas expressing the homology and cohomology of a space with arbitrary coefficients as functors of the integral homology of the space. Although these new functors do not distinguish between spaces not already distinguished by the integral homology functor, it is nonetheless important to consider them, as it frequently happens that the most natural functor to apply in a given geometrical problem is determined by the problem itself and need not be the integral homology functor. For example, in the obstruction theory developed in Chapter Eight we shall be led to the cohomology of a space with coefficients in the homotopy groups of another space.

A further consideration is that the cohomology of a space has a multiplicative structure in addition to its additive structure, which makes cohomology a more powerful tool than homology. We shall present some applications of this added multiplication structure, the most important of which is the study of the homology properties of fiber bundles, where we establish the exactness of the Thom-Gysin sequence of a sphere bundle.

At the end of the chapter is a brief discussion of cohomology operations. These are natural transformations between two cohomology functors and strengthen even further the applicability of cohomology as a tool. We shall define the particular set of cohomology operations known as the Steenrod squares and establish their basic properties.

Sections 5.1 and 5.2 are devoted to homology with general coefficients and to the universal-coefficient formula for homology. Section 5.3 deals with the tensor product of two chain complexes and contains a proof of the Künneth formula expressing the homology of the tensor product as a functor of the homology of the factor complexes. This is applied geometrically to express the homology of a product space in terms of the homology of its factors.

Sections 5.4 and 5.5 contain the dual concepts of cochain complex and cohomology and the appropriate universal-coefficient formulas for them. In Sec. 5.6 the cup and cap products are defined, the cup product being the multiplicative structure in cohomology mentioned previously, and the cap product being a dual involving cohomology and homology together. These products are used in Sec. 5.7 to study the homology and cohomology of fiber bundles. We establish the Leray-Hirsch theorem, which asserts that certain fiber bundles have homology and cohomology which are additively isomorphic to the homology and cohomology of the corresponding product of the base and the fiber.

Section 5.8 is devoted to a study of the cohomology algebra. The exactness of the Thom-Gysin sequence is used to compute the cohomology algebra of projective spaces, and this, in turn, is used to prove the Borsuk-Ulam theorem. There is also a discussion of the structure of Hopf algebras, which arise in considering the cohomology of an H space. In Sec. 5.9 the Steenrod squares are defined and their elementary properties are proved. They will be applied later.

1 HOMOLOGY WITH COEFFICIENTS

In this section we shall extend the concepts dealing with chain complexes to the case where the chain groups are modules over a ring. The tensor product of such a chain complex with a fixed module is another chain complex, and its graded homology module is a functor of the original chain complex and the fixed module. These homology modules have properties analogous to those established in the last chapter for complexes of abelian groups. The section closes with the definition of a homology theory with an arbitrary coefficient module. This is analogous to the concept of homology theory (which has integral coefficients) introduced in the last chapter.

Throughout this section R will denote a commutative ring with a unit. We consider R modules and homomorphisms between them. A *chain complex over* R, $C = \{C_q,\partial_q\}$ consists of a sequence of R modules C_q and homomorphisms $\partial_q\colon C_q \to C_{q-1}$ such that $\partial_q\partial_{q+1} = 0$ for all q. There is then a graded homology module

$$H(C) = \{H_q(C) = \ker \partial_q/\mathrm{im}\ \partial_{q+1}\}$$

The concepts of chain maps and chain homotopies can be defined for chain complexes over R, and the results about chain complexes of abelian groups generalize in a straightforward fashion to chain complexes over R. In particular, on the category of short exact sequences of chain complexes over R,

$$0 \to C' \to C \to C'' \to 0$$

there is a functorial connecting homomorphism

$$\partial_*\colon H_q(C'') \to H_{q-1}(C')$$

and a functorial exact sequence

$$\cdots \xrightarrow{\partial_*} H_q(C') \to H_q(C) \to H_q(C'') \xrightarrow{\partial_*} H_{q-1}(C') \to \cdots$$

If C is a chain complex over R and G' is an R module, an *augmentation of* C *over* G' is an epimorphism $\varepsilon\colon C_0 \to G'$ such that $\varepsilon \circ \partial_1 = 0$. An *augmented chain complex over* G' consists of a nonnegative chain complex C and an augmentation of C over G'.

If $C = \{C_q,\partial_q\}$ is a chain complex over R and G is an R module, then $C \otimes G = \{C_q \otimes G,\ \partial_q \otimes 1\}$ is also a chain complex over R, and if C is augmented over G', then $C \otimes G$ is augmented over $G' \otimes G$. The graded homology module $H(C \otimes G)$ is called the *homology module of* C *with coefficients* G and is denoted by $H(C;G)$. If $\tau\colon C \to C'$ is a chain map, $\tau \otimes 1\colon C \otimes G \to C' \otimes G$ is also a chain map, and $\tau_*\colon H(C;G) \to H(C';G)$ denotes the homomorphism induced by $\tau \otimes 1$. Given a homomorphism $\varphi\colon G \to G'$, there is a chain map $1 \otimes \varphi\colon C \otimes G \to C \otimes G'$ inducing a homomorphism

$$\varphi_*\colon H(C;G) \to H(C;G')$$

These remarks are summarized in the following statement.

1 THEOREM *There is a covariant functor of two arguments from the category of chain complexes over R and the category of R modules to the category of graded R modules which assigns to a chain complex C and module G the homology module of C with coefficients G.* ∎

Note that if $c \in C_q$ is a cycle of C and $g \in G$, then $c \otimes g \in C_q \otimes G$ is a cycle of $C \otimes G$, and if c is a boundary, so is $c \otimes g$. Therefore there is a bilinear map

$$H_q(C) \times G \to H_q(C;G)$$

which assigns to $(\{c\},g)$ the homology class $\{c \otimes g\}$. This corresponds to a homomorphism

$$\mu\colon H(C) \otimes G \to H(C;G)$$

such that $\mu(\{c\} \otimes g) = \{c \otimes g\}$ for $c \in Z(C)$. The homomorphism μ is easily verified to be a natural transformation on the product of the category of chain complexes with the category of modules.

If C is a chain complex over $\mathbf{Z}$ and G is an R module, then $C \underset{\mathbf{Z}}{\otimes} G$ is a chain complex over R. It follows from theorem 4.5 in the Introduction that the homology module over $\mathbf{Z}$ of C with coefficients G is isomorphic, as a graded R module, to the homology module over R of $C \underset{\mathbf{Z}}{\otimes} R$ with coefficients G.

2 EXAMPLE Let $C(K)$ denote the oriented chain complex of the simplicial complex K. Given an abelian group G and a simplicial pair (K,L), the *oriented homology group of* (K,L) *with coefficients* G, denoted by $H(K,L;\, G)$, is defined to be the graded homology group of $[C(K)/C(L)] \otimes G$ (which is augmented over $\mathbf{Z} \otimes G \approx G$). Then $H(K,L;\, G)$ is a covariant functor of two arguments from the category of simplicial pairs and the category of abelian groups to the category of graded abelian groups. If G is also an R module, $H(K,L;\, G)$ is a graded R module. Similar remarks apply to the ordered chain complex $\Delta(K)/\Delta(L)$.

3 EXAMPLE If (X,A) is a topological pair and G is an abelian group, the *singular homology group of* (X,A) *with coefficients* G, denoted by $H(X,A;\, G)$, is defined to be the graded homology group of $[\Delta(X)/\Delta(A)] \otimes G$ (which is augmented over G). It is a covariant functor of two arguments from the category of topological pairs and the category of abelian groups to the category of graded abelian groups. If G is an R module, $H(X,A;\, G)$ is a graded R module.

Because the ring R is commutative, there is a canonical isomorphism $G \otimes G' \approx G' \otimes G$ for R modules G and G'. Therefore, if C is a chain complex over R, $G \otimes C$ is canonically isomorphic to $C \otimes G$. Hence no new homology modules are obtained from $G \otimes C$.

We recall some general properties of tensor products which will be important in the next section.

4 LEMMA *The tensor product of two epimorphisms is an epimorphism.*

PROOF Let $\alpha\colon A \to A''$ and $\beta\colon B \to B''$ be epimorphisms. $A'' \otimes B''$ is generated by elements of the form $a'' \otimes b''$, where $a'' \in A''$ and $b'' \in B''$. Since α and β are epimorphisms, $A'' \otimes B''$ is generated by elements of the form

$\alpha(a) \otimes \beta(b)$, where $a \in A$ and $b \in B$. Since $(\alpha \otimes \beta)(a \otimes b) = \alpha(a) \otimes \beta(b)$, $A'' \otimes B''$ is generated by $(\alpha \otimes \beta)(A \otimes B)$, showing that $\alpha \otimes \beta$ is an epimorphism. ■

In general, it is not true that the tensor product of two monomorphisms is a monomorphism (see example 7 below). The following lemma shows that something can be said about the kernel of $\alpha \otimes \beta$ when α and β are epimorphisms.

5 LEMMA *If α and β are epimorphisms, the kernel of $\alpha \otimes \beta$ is generated by elements of the form $a \otimes b$, where $a \in \ker \alpha$ or $b \in \ker \beta$.*

PROOF Let $\alpha: A \to A''$ and $\beta: B \to B''$ be epimorphisms and let D be the submodule of $A \otimes B$ generated by elements of the form $a \otimes b$, where $a \in \ker \alpha$ or $b \in \ker \beta$. Let $p: A \otimes B \to (A \otimes B)/D$ be the projection. There is a well-defined bilinear map

$$A'' \times B'' \to (A \otimes B)/D$$

sending (a'',b'') to $p(a \otimes b)$, where $a \in A$ and $b \in B$ are chosen so that $\alpha(a) = a''$ and $\beta(b) = b''$. This bilinear map corresponds to a homomorphism

$$\psi: A'' \otimes B'' \to (A \otimes B)/D$$

such that $\psi(a'' \otimes b'') = p(a \otimes b)$, where $\alpha(a) = a''$ and $\beta(b) = b''$. It is then obvious that p equals the composite

$$A \otimes B \xrightarrow{\alpha \otimes \beta} A'' \otimes B'' \xrightarrow{\psi} (A \otimes B)/D$$

This shows that $\ker (\alpha \otimes \beta) \subset D$. The reverse inclusion is evident, showing that $\ker (\alpha \otimes \beta) = D$. ■

6 COROLLARY *Given an exact sequence*

$$A' \to A \to A'' \to 0$$

and given a module B, there is an exact sequence

$$A' \otimes B \to A \otimes B \to A'' \otimes B \to 0$$

PROOF It follows from lemma 4 that $A \otimes B \to A'' \otimes B$ is an epimorphism, so the sequence is exact at $A'' \otimes B$. If $\bar{A} \subset A$ is the image of $A' \to A$, then, by lemma 4, $A' \otimes B \to \bar{A} \otimes B$ is an epimorphism. Because $\bar{A}$ is also the kernel of $A \to A''$, it follows from lemma 5 that the kernel of $A \otimes B \to A'' \otimes B$ is the image of $\bar{A} \otimes B \to A \otimes B$. Therefore the sequence is exact at $A \otimes B$. ■

If the original sequence is assumed to be a short exact sequence, it need not be true that the tensor-product sequence is a short exact sequence. We present an example to illustrate this.

7 EXAMPLE Over $\mathbf{Z}$, consider the short exact sequence

$$0 \to \mathbf{Z} \xrightarrow{\alpha} \mathbf{Z} \xrightarrow{\beta} \mathbf{Z}_2 \to 0$$

where $\alpha(1) = 2$ and $\beta(1)$ is a generator $\bar{1}$ of $\mathbf{Z}_2$. The tensor product of this sequence with $\mathbf{Z}_2$ is not a short exact sequence because $\alpha \otimes 1: \mathbf{Z} \otimes \mathbf{Z}_2 \to \mathbf{Z} \otimes \mathbf{Z}_2$ is not a monomorphism $[\mathbf{Z} \otimes \mathbf{Z}_2 \approx \mathbf{Z}_2 \neq 0$, but $(\alpha \otimes 1)(1 \otimes \bar{1}) = 2 \otimes \bar{1} = 1 \otimes 2 \cdot \bar{1} = 0]$.

8 THEOREM *The tensor-product functor commutes with direct sums.*

PROOF Assume $A = \bigoplus A_j$ and consider the bilinear map $A \times B \to \bigoplus (A_j \otimes B)$ sending $(\Sigma\, a_j, b)$ to $\Sigma\, (a_j \otimes b)$ and the homomorphisms $A_j \otimes B \to A \otimes B$ for all j. By the characteristic properties of tensor product and direct sum, there are commutative triangles

$$\begin{tikzcd} A \times B \arrow[d] \arrow[dr] & \\ A \otimes B \arrow[r, "\varphi"] & \bigoplus (A_j \otimes B) \end{tikzcd} \qquad \begin{tikzcd} & A_j \otimes B \arrow[dl] \arrow[d] \\ A \otimes B & \bigoplus (A_j \otimes B) \arrow[l, "\psi"'] \end{tikzcd}$$

Clearly, the maps φ and ψ are inverses, showing that $A \otimes B \approx \bigoplus (A_j \otimes B)$. If, also, $B = \bigoplus B_k$, then similarly,

$$A \otimes B \approx \bigoplus_{j,k} A_j \otimes B_k \quad \blacksquare$$

9 THEOREM *The tensor-product functor commutes with direct limits.*

PROOF Let $A = \lim_{\to} \{A^\alpha\}$ and consider the bilinear map $A \times B \to \lim_{\to} \{A^\alpha \otimes B\}$ sending $(\{a\},b)$ to $\{a \otimes b\}$ for $a \in A^\alpha$ and the homomorphisms $A^\alpha \otimes B \to A \otimes B$ for all α. By the characteristic properties of tensor product and direct limit, there are commutative triangles

$$\begin{tikzcd} A \times B \arrow[d] \arrow[dr] & \\ A \otimes B \arrow[r, "\varphi"] & \lim_{\to} \{A^\alpha \otimes B\} \end{tikzcd} \qquad \begin{tikzcd} & A^\alpha \otimes B \arrow[dl] \arrow[d] \\ A \otimes B & \lim_{\to} \{A^\alpha \otimes B\} \arrow[l, "\psi"'] \end{tikzcd}$$

Clearly, φ and ψ are inverses, showing that $A \otimes B \approx \lim_{\to} \{A^\alpha \otimes B\}$. If, also, $B = \lim_{\to} \{B^\beta\}$, then similarly, $A \otimes B \approx \lim_{\to} \{A^\alpha \otimes B^\beta\}$. $\blacksquare$

We now consider a special class of short exact sequences. These sequences have the property that their tensor product with any module is again exact. A short exact sequence

$$0 \to A' \xrightarrow{\alpha} A \xrightarrow{\beta} A'' \to 0$$

is said to be *split* if β has a right inverse (that is, if there exists a homomorphism $\beta': A'' \to A$ such that $\beta \circ \beta' = 1_{A''}$). We also say that the sequence *splits*.

10 EXAMPLE Any short exact sequence $0 \to A' \to A \xrightarrow{\beta} A'' \to 0$ with A'' free is split. To see this, let $\{a_j''\}$ be a basis for A'' and for each j choose $a_j \in A$ so that $\beta(a_j) = a_j''$. Let $\beta': A'' \to A$ be the homomorphism such that $\beta'(a_j'') = a_j$ for all j. Then β' is a right inverse of β.

11 LEMMA *Given a short exact sequence*

$$0 \to A' \xrightarrow{\alpha} A \xrightarrow{\beta} A'' \to 0$$

define $A' \xrightarrow{i} A' \oplus A'' \xrightarrow{p} A''$ *by* $i(a') = (a',0)$ *and* $p(a',a'') = a''$. *Then the following are equivalent:*

(*a*) *The sequence is split.*

(*b*) *There is a commutative diagram*

$$\begin{tikzcd} & A \arrow[dr, "\beta"] & \\ A' \arrow[ur, "\alpha"] \arrow[dr, "i"'] & & A'' \\ & A' \oplus A'' \arrow[uu, "\gamma'"] \arrow[ur, "p"'] & \end{tikzcd}$$

(*c*) *There is a commutative diagram*

$$\begin{tikzcd} & A \arrow[dr, "\beta"] \arrow[dd, "\gamma"] & \\ A' \arrow[ur, "\alpha"] \arrow[dr, "i"'] & & A'' \\ & A' \oplus A'' \arrow[ur, "p"'] & \end{tikzcd}$$

(*d*) α *has a left inverse.*

PROOF If β': $A'' \to A$ is a right inverse of β, let γ': $A' \oplus A'' \to A$ be defined by $\gamma'(a',a'') = \alpha(a') + \beta'(a'')$. Then γ' has the desired properties. Conversely, given γ', define β': $A'' \to A$ by $\beta'(a'') = \gamma'(0,a'')$. Then β' is a right inverse of β, so the sequence is split. Therefore (*a*) is equivalent to (*b*). A similar argument shows that (*c*) is equivalent to (*d*). It follows from the five lemma that in the diagram of (*b*) [or (*c*)], γ' [or γ] is necessarily an isomorphism. Therefore (*b*) is equivalent to (*c*) with γ' equal to γ^{-1}. ∎

12 COROLLARY *Given a split short exact sequence*

$$0 \to A' \xrightarrow{\alpha} A \to A'' \to 0$$

and given a module B, the sequence

$$0 \to A' \otimes B \xrightarrow{\alpha \otimes 1} A \otimes B \to A'' \otimes B \to 0$$

is a split short exact sequence.

PROOF By corollary 6 and lemma 11 we need only show that $\alpha \otimes 1$ has a left inverse. By lemma 11, α has a left inverse α'. Then $\alpha' \otimes 1$ is a left inverse of $\alpha \otimes 1$. ∎

In case $0 \to C' \to C \to C'' \to 0$ is a split short exact sequence of chain complexes, it follows from corollary 12 that for any module G the sequence

$$0 \to C' \otimes G \to C \otimes G \to C'' \otimes G \to 0$$

is a short exact sequence of chain complexes. This short exact sequence gives rise to an exact homology sequence, and we obtain the next result.

13 THEOREM *Given a split short exact sequence of chain complexes*

$$0 \to C' \to C \to C'' \to 0$$

and given a module G, there is a functorial exact homology sequence

$$\cdots \to H_q(C';G) \to H_q(C;G) \to H_q(C'';G) \to H_{q-1}(C';G) \to \cdots \quad \blacksquare$$

This implies the exactness of the singular homology sequence (and reduced homology sequence) of a pair with arbitrary coefficients. Similarly, there is an exact sequence of a triple with arbitrary coefficients. All these sequences (except the reduced sequence of a pair) are consequences of the exactness of the relative Mayer-Vietoris sequence, which we now establish. If $\{(X_1,A_1), (X_2,A_2)\}$ is an excisive couple of pairs in a topological space, the short exact sequence of singular chain complexes

$$0 \to \Delta(X_1 \cap X_2)/\Delta(A_1 \cap A_2) \to$$
$$\Delta(X_1)/\Delta(A_1) \oplus \Delta(X_2)/\Delta(A_2) \to [\Delta(X_1) + \Delta(X_2)]/[\Delta(A_1) + \Delta(A_2)] \to 0$$

is split [by example 10, because $[\Delta(X_1) + \Delta(X_2)]/[\Delta(A_1) + \Delta(A_2)]$ is a free abelian group]. Therefore we obtain the following result.

14 COROLLARY *If $\{(X_1,A_1), (X_2,A_2)\}$ is an excisive couple of pairs in a space and G is an R module, there is an exact relative Mayer-Vietoris sequence of $\{(X_1,A_1), (X_2,A_2)\}$ with coefficients G.* $\blacksquare$

If G is fixed, the singular homology of (X,A) with coefficients G satisfies all the axioms of homology theory except the dimension axiom (all of them are easily seen to hold except exactness, which follows from corollary 14). If P is a one-point space, there is a functorial isomorphism $H_0(P;G) \approx G$. This leads to the following definition.

Let G be an R module. A *homology theory with coefficients* G consists of a covariant functor H from the category of topological pairs to graded R modules and a natural transformation ∂: $H(X,A) \to H(A)$ of degree -1 satisfying the homotopy, exactness, and excision axioms, and satisfying the following form of the dimension axiom: On the category of one-point spaces there is a natural equivalence of H with the constant functor which assigns to every one-point space the graded module which is trivial for degrees other than 0 and equal to G in degree 0. A homology theory with coefficients $\mathbf{Z}$ is called an *integral homology theory.* An integral homology theory is the same as a homology theory as defined in Sec. 4.8.

Singular homology with coefficients G is an example of a homology theory with coefficients G. The uniqueness theorem 4.8.10 is valid for homology theories with coefficients.

In the next section we shall show how the singular homology modules with coefficients are determined by the integral singular homology groups.

2 THE UNIVERSAL-COEFFICIENT THEOREM FOR HOMOLOGY

In order to express $H(C;G)$ in terms of $H(C)$ and G, it is necessary to introduce certain functors of modules that are associated to the tensor-product functor. This section contains a definition of these functors, and a study of them in the special case of a principal ideal domain. This leads to the universal-coefficient theorem. In the next section these new functors will enter in a description of the homology of a product space.

Let A be an R module. A *resolution of* A (*over* R) is an exact sequence

$$\cdots \to C_n \xrightarrow{\partial_n} \cdots \to C_1 \xrightarrow{\partial_1} C_0 \xrightarrow{\varepsilon} A \to 0$$

If, in addition, each C_q is a free R module, the resolution is said to be *free*. Thus a resolution of A consists of a chain complex $C = \{C_q,\partial_q\}$ over R which is augmented over A and is such that $\tilde{C}$ is acyclic. The resolution is free if and only if the chain complex C is free.

Any R module A has free resolutions. In fact, given an R module B, let $F(B)$ be the free R module generated by the elements of B and let $F(B) \to B$ be the canonical map. The *canonical* free resolution of A is the following resolution (defined inductively):

$$\cdots \to F(\ker \partial_q) \xrightarrow{\partial_{q+1}} F(\ker \partial_{q-1}) \xrightarrow{\partial_q} \cdots \to F(\ker \varepsilon) \xrightarrow{\partial_1} F(A) \xrightarrow{\varepsilon} A \to 0$$

The method of acyclic models applies to chain complexes over R and, when applied to a category consisting of a single object and single morphism, implies the following result.

1 THEOREM *Let C be a free nonnegative chain complex augmented over A and let C' be a resolution of A'. Any homomorphism $\varphi: A \to A'$ extends to a chain map*

$$\begin{array}{ccccccccc}
\cdots \to & C_{q+1} & \to & C_q & \to \cdots \to & C_0 & \xrightarrow{\varepsilon} & A & \to 0 \\
& \downarrow & & \downarrow & & \downarrow & & \downarrow\varphi & \\
\cdots \to & C'_{q+1} & \to & C'_q & \to \cdots \to & C'_0 & \xrightarrow{\varepsilon'} & A' & \to 0
\end{array}$$

preserving augmentations, and two such chain maps are chain homotopic. ■

Specializing to the case $\varphi = 1_A: A \subset A$, we obtain the next result.

2 COROLLARY *If C and C' are free resolutions of A, then C and C' are canonically chain-equivalent chain complexes.* ■

For modules A and B and a free resolution C of A, it follows from corollary 2 that the graded module $H(C;B)$ depends only on A and B. Let C be the canonical free resolution of A. For $q \geq 0$ we define the *qth torsion product* $\operatorname{Tor}_q(A,B) = H_q(C;B)$. It is a covariant functor of A and of B. From the short exact sequence

$$0 \to \partial_1 C_1 \to C_0 \xrightarrow{\varepsilon} A \to 0$$

it follows from corollary 5.1.6 that there is an exact sequence

$$\partial_1 C_1 \otimes B \to C_0 \otimes B \xrightarrow{\varepsilon \otimes 1} A \otimes B \to 0$$

By definition, $\mathrm{Tor}_0\,(A,B)$ is the zeroth homology module of the chain complex

$$\cdots \to C_2 \otimes B \to C_1 \otimes B \xrightarrow{\partial_1 \otimes 1} C_0 \otimes B \to 0$$

Hence $\mathrm{Tor}_0\,(A,B) = (C_0 \otimes B)/\mathrm{im}\,(\partial_1 \otimes 1)$. By the above exact sequence,

$$\mathrm{im}\,(\partial_1 \otimes 1) = \mathrm{im}\,(\partial_1 C_1 \otimes B \to C_0 \otimes B) = \ker\,(\varepsilon \otimes 1)$$

Therefore

$$\mathrm{Tor}_0\,(A,B) = (C_0 \otimes B)/\ker\,(\varepsilon \otimes 1) \approx A \otimes B$$

and so $\mathrm{Tor}_0\,(A,B)$ is naturally equivalent to $A \otimes B$.

All the previous remarks are valid for any commutative ring with a unit. For the remainder of this section we specialize to the case where R is a principal ideal domain. Over a principal ideal domain any submodule of a free module is free. Therefore any module A has a short free resolution of the form

$$0 \to C_1 \to C_0 \to A \to 0$$

(simply let $C_0 = F(A)$ and $C_1 = \ker\,[F(A) \to A]$). Such a short free resolution of A is the same as a free presentation of A. Because there exist short free resolutions, $\mathrm{Tor}_q\,(A,B) = 0$ if $q > 1$. We define the *torsion product* $A * B$ to equal $\mathrm{Tor}_1\,(A,B)$. It is characterized by the property that, given any free presentation of A,

$$0 \to C_1 \to C_0 \to A \to 0$$

there is an exact sequence

$$0 \to A * B \to C_1 \otimes B \to C_0 \otimes B \to A \otimes B \to 0$$

In fact, $A * B \approx H_1(C \otimes B) = \ker\,(C_1 \otimes B \to C_0 \otimes B)$, since $C_2 \otimes B = 0$.

The torsion product is a covariant functor of each of its arguments. Because the tensor product commutes with direct sums and direct limits (by theorems 5.1.8 and 5.1.9) and the direct limit of exact sequences is exact (by theorem 4.5.7), the torsion product also commutes with direct sums and direct limits. Its name derives from the fact that it depends only on the torsion submodules of A and B (see corollary 11 below).

3 EXAMPLE If A is free, it has the free presentation

$$0 \to 0 \to A \to A \to 0$$

from which we see that $A * B = 0$ for any B.

4 EXAMPLE If A is the cyclic R module whose annihilating ideal is generated by an element $v \in R$, then $A \approx R/vR$ and there is a free presentation of A

$$0 \to R \xrightarrow{\alpha} R \to A \to 0$$

in which $\alpha(v') = vv'$ for $v' \in R$. For any module B there is an isomorphism $R \otimes B \approx B$ sending $1 \otimes b$ to b. Under this isomorphism, the map $\alpha \otimes 1: R \otimes B \to R \otimes B$ corresponds to $\alpha': B \to B$, where $\alpha'(b) = vb$ for $b \in B$. Therefore ker α' is the submodule of B annihilated by v, and so

$$(R/vR) * B \approx \{b \in B \mid vb = 0\}$$

The above examples suffice to compute $A * B$ for a finitely generated module A (because of the structure theorem 4.14 in the Introduction). This theoretically determines $A * B$ for arbitrary A, because any A is the direct limit of its finitely generated submodules (see theorem 4.2 in the Introduction) and the torsion product commutes with direct limits.

5 LEMMA *If A or B is torsion free, then $A * B = 0$.*

PROOF Because the torsion product commutes with direct limits, it suffices to consider the case where A and B are finitely generated, in which case being torsion free is equivalent to being free. If A is free, the result follows from example 3. If B is free and finitely generated, it is isomorphic to a direct sum of n copies of R. If

$$0 \to C_1 \to C_0 \to A \to 0$$

is a free presentation of A, then $C_1 \otimes B \to C_0 \otimes B \to A \otimes B \to 0$ is isomorphic to a direct sum of n copies of the sequence $C_1 \otimes R \to C_0 \otimes R \to A \otimes R \to 0$. Since $C_1 \otimes R \to C_0 \otimes R$ is a monomorphism, so is $C_1 \otimes B \to C_0 \otimes B$, and $A * B = 0$. ■

It follows that if R is a field, then $A * B = 0$ for all modules A and B. The following result is proved similarly by proving it first for finitely generated modules (where being torsion free is equivalent to being free) and taking direct limits to obtain the result for arbitrary modules.

6 LEMMA *Given a short exact sequence of modules*

$$0 \to A' \to A \to A'' \to 0$$

and given a module B, if A'' or B is torsion free, there is a short exact sequence

$$0 \to A' \otimes B \to A \otimes B \to A'' \otimes B \to 0$$

PROOF As remarked above, it suffices to prove the result if A'' or B is free and finitely generated. If A'' is free, the original sequence splits, by example 5.1.10, and the result follows from corollary 5.1.12. If B is free and finitely generated, the map $A' \otimes B \to A \otimes B$ is a finite direct sum of copies of $A' \otimes R \to A \otimes R$, and hence a monomorphism. The result follows from this and corollary 5.1.6. ■

We use this result to obtain an exact sequence of homology corresponding to a short exact sequence of coefficient modules.

7 THEOREM *On the product category of torsion-free chain complexes C and short exact sequences of modules*

$$0 \to G' \xrightarrow{\varphi} G \xrightarrow{\psi} G'' \to 0$$

there is a natural connecting homomorphism

$$\beta\colon H(C;G'') \to H(C;G')$$

of degree -1 *and a functorial exact sequence*

$$\cdots \to H_q(C;G') \xrightarrow{\varphi_*} H_q(C;G) \xrightarrow{\psi_*} H_q(C;G'') \xrightarrow{\beta} H_{q-1}(C;G') \to \cdots$$

PROOF By lemma 6, there is a short exact sequence of chain complexes

$$0 \to C \otimes G' \xrightarrow{1 \otimes \varphi} C \otimes G \xrightarrow{1 \otimes \psi} C \otimes G'' \to 0$$

Since this is functorial in C and in the exact coefficient sequence, the result follows from theorem 4.5.4. ■

The connecting homomorphism β occurring in theorem 7 is called the *Bockstein homology homomorphism* corresponding to the coefficient sequence $0 \to G' \xrightarrow{\varphi} G \xrightarrow{\psi} G' \to 0$. Theorem 7 remains valid over an arbitrary commutative ring R with a unit if C is assumed to be a free chain complex over R.

Let C be a chain complex over R and let G be an R module. Recall the homomorphism $\mu\colon H(C) \otimes G \to H(C;G)$ defined in the last section. This homomorphism enters in the following *universal-coefficient theorem for homology.*

8 THEOREM *Let C be a free chain complex and let G be a module. There is a functorial short exact sequence*

$$0 \to H_q(C) \otimes G \xrightarrow{\mu} H_q(C \otimes G) \to H_{q-1}(C) * G \to 0$$

and this sequence is split.

PROOF Let Z be the subcomplex of C defined by $Z_q = Z_q(C)$ with trivial boundary operator and let B be the complex defined by $B_q = B_{q-1}(C)$ with trivial boundary operator. Both B and Z are free chain complexes and there is a short exact sequence

$$0 \to Z \xrightarrow{\alpha} C \xrightarrow{\beta} B \to 0$$

where $\alpha_q(z) = z$ for $z \in Z_q$ and $\beta_q(c) = \partial_q c$ for $c \in C_q$. Since B is a free complex, this short exact sequence is split. By theorem 5.1.13, there is an exact sequence

$$\cdots \to H_q(Z;G) \xrightarrow{\alpha_*} H_q(C;G) \xrightarrow{\beta_*} H_q(B;G) \xrightarrow{\partial_*} H_{q-1}(Z;G) \to \cdots$$

where $\partial_* \{b\} = \{\alpha_{q-1}^{-1} \partial_q \partial_q^{-1} b\} = \{\alpha_{q-1}^{-1}(b)\}$ for $b \in B_{q-1}$. Since Z and B have trivial boundary operators, so do $Z \otimes G$ and $B \otimes G$. Therefore $H_q(Z;G) = Z_q \otimes G$ and $H_q(B;G) = B_q \otimes G = B_{q-1}(C) \otimes G$, and the above exact sequence becomes

$$\cdots \to B_q(C) \otimes G \xrightarrow{\gamma_q \otimes 1} Z_q(C) \otimes G \to H_q(C;G) \to B_{q-1}(C) \otimes G \xrightarrow{\gamma_{q-1} \otimes 1} Z_{q-1}(C) \otimes G \to \cdots$$

where $\gamma_q\colon B_q(C) \subset Z_q(C)$. From the exactness of this sequence we obtain a short exact sequence

$$0 \to \operatorname{coker}(\gamma_q \otimes 1) \to H_q(C;G) \to \ker(\gamma_{q-1} \otimes 1) \to 0$$

and it only remains to interpret the modules on either side of $H_q(C;G)$.

Since $Z_q(C)$ is free, the short exact sequence

$$0 \to B_q(C) \xrightarrow{\gamma_q} Z_q(C) \to H_q(C) \to 0$$

is a free presentation of $H_q(C)$. By the characteristic property of the torsion product, there is an exact sequence

$$0 \to H_q(C) * G \to B_q(C) \otimes G \xrightarrow{\gamma_q \otimes 1} Z_q(C) \otimes G \to H_q(C) \otimes G \to 0$$

Therefore $\operatorname{coker}(\gamma_q \otimes 1) \approx H_q(C) \otimes G$ and $\ker(\gamma_q \otimes 1) \approx H_q(C) * G$. Substituting these into the short exact sequence above yields the short exact sequence

$$0 \to H_q(C) \otimes G \to H_q(C;G) \to H_{q-1}(C) * G \to 0$$

It is easily verified by checking the definitions that the homomorphism $H_q(C) \otimes G \to H_q(C;G)$ is equal to μ.

If $\tau\colon C \to C'$ is a chain map, τ defines a commutative diagram

$$\begin{array}{ccccccccc} 0 & \to & Z & \xrightarrow{\alpha} & C & \xrightarrow{\beta} & B & \to & 0 \\ & & {\scriptstyle\tau'}\downarrow & & {\scriptstyle\tau}\downarrow & & \downarrow{\scriptstyle\tau''} & & \\ 0 & \to & Z' & \xrightarrow{\alpha'} & C' & \xrightarrow{\beta'} & B' & \to & 0 \end{array}$$

from which we obtain the commutative diagram

$$\begin{array}{ccccccccc} 0 & \to & H_q(C) \otimes G & \xrightarrow{\mu} & H_q(C;G) & \to & H_{q-1}(C) * G & \to & 0 \\ & & {\scriptstyle\tau_* \otimes 1}\downarrow & & \downarrow{\scriptstyle\tau_*} & & \downarrow{\scriptstyle\tau_* * 1} & & \\ 0 & \to & H_q(C') \otimes G & \xrightarrow{\mu} & H_q(C';G) & \to & H_{q-1}(C') * G & \to & 0 \end{array}$$

Therefore the short exact sequence for $H_q(C;G)$ is functorial.

We now prove that the short exact sequence is split (but is not functorially split). Because $B_{q-1}(C)$ is free and $\partial_q C_q = B_{q-1}(C)$, there exist homomorphisms $h_q\colon B_{q-1}(C) \to C_q$ such that $\partial_q h_q = 1$. Then

$$h_q \otimes 1\colon B_{q-1}(C) \otimes G \to C_q \otimes G$$

maps the kernel of $\gamma_{q-1} \otimes 1$ into cycles of $C_q \otimes G$ and induces a homomorphism $H_{q-1}(C) * G \to H_q(C;G)$ which is a right inverse of the homomorphism $H_q(C;G) \to H_{q-1}(C) * G$ of the short exact sequence in the theorem. ∎

We can use this result to establish some properties of the torsion product, beginning with the following *six-term exact sequence* connecting the tensor and torsion products.

9 COROLLARY *Let $0 \to B' \xrightarrow{\alpha'} B \xrightarrow{\beta'} B'' \to 0$ be a short exact sequence of modules and let A be a module. There is an exact sequence*

$$0 \to A * B' \xrightarrow{1 * \alpha'} A * B \xrightarrow{1 * \beta'} A * B'' \to$$
$$A \otimes B' \xrightarrow{1 \otimes \alpha'} A \otimes B \xrightarrow{1 \otimes \beta'} A \otimes B'' \to 0$$

PROOF Let $0 \to C_1 \to C_0 \to A \to 0$ be a free presentation of A and let C be the corresponding free chain complex obtained by adding trivial groups on both sides. Since C is free, it follows from lemma 6 that there is a short exact sequence of chain complexes

$$0 \to C \otimes B' \xrightarrow{1 \otimes \alpha'} C \otimes B \xrightarrow{1 \otimes \beta'} C \otimes B'' \to 0$$

Because $H_q(C) = 0$ if $q \neq 0$ and $H_0(C) = A$, the homology sequence of the above short exact sequence of chain complexes (interpreted by means of theorem 8) gives the desired exact sequence. ∎

This yields the commutativity of the torsion product.

10 COROLLARY *There is a functorial isomorphism*

$$A * B \approx B * A$$

PROOF Let $0 \to C_1 \to C_0 \to B \to 0$ be a free presentation of B. By corollary 9, there is an exact sequence

$$0 \to A * C_1 \to A * C_0 \to A * B \to A \otimes C_1 \to A \otimes C_0 \to A \otimes B \to 0$$

Since C_0 is free, it follows from lemma 5 that $A * C_0 = 0$, and there is an exact sequence

$$0 \to A * B \to A \otimes C_1 \to A \otimes C_0 \to A \otimes B \to 0$$

By the characteristic property of $B * A$, there is an exact sequence

$$0 \to B * A \to C_1 \otimes A \to C_0 \otimes A \to B \otimes A \to 0$$

The functorial isomorphism $A * B \approx B * A$ then results by chasing in the commutative diagram

$$\begin{array}{ccccccccc}
0 \to & A * B & \to & A \otimes C_1 & \to & A \otimes C_0 & \to & A \otimes B & \to 0 \\
& & & \downarrow\approx & & \downarrow\approx & & \downarrow\approx & \\
0 \to & B * A & \to & C_1 \otimes A & \to & C_0 \otimes A & \to & B \otimes A & \to 0
\end{array}$$

in which the vertical maps are the functorial isomorphisms expressing the